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Real Projective Space

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Introduction

By the 1860's, the work of Èvariste Galois, and in particular the idea of a permutation group, had begun to be known to mathematicians in Germany. The young Felix Klein recognized how this idea could be used to organize and describe the various geometries which were being studied at the time, and presented his thoughts as his inaugural address at the University of Erlangen in 1872. So began the famous "Erlanger Programm" which was to have long lasting influence on the development of mathematics, and not only in the field of geometry.

Set theory had not yet been developed as we know it today. Klein called the domains on which the permutations acted "Mannigfaltigkeiten", or manifolds. He apparently used the word to represent the general conception of a set exhibiting some sort of geometry. The permutation group represented the ways the manifold could be transformed without changing the intrinsic geometric properties of the objects being transformed. In the language of contemporary mathematics his program could be described as follows:

(1) Begin with a set X exhibiting a certain intrinsic property or properties;

(2) Take a group of permutations of X and select out the subgroup $\mathfrak{X}$ consisting precisely of those permutations which preserve the property or properties described in (1);

(3) Find other properties of X which are preserved by the elements of $\mathfrak{X}$.

There is a symbiotic relationship between the set X which represents the manifestation of the geometry and the family $\mathfrak{X}$, which represents the various ways X can be observed without causing it to lose its defining qualities.

Many of the geometries which were being investigated *circa* 1872 were related to projective geometry, and for the permutation group $\mathfrak{X}$, Klein took the group of all projective transformations of projective space.

In the present treatment of projective space we show how projective space can be defined using the spirit of the Erlanger Programm, but entirely within the context of a set paired with a group of permutations, there being no properties of the set or the group introduced *a priori*. These are replaced by three postulates imposed on the relationship between the set and the group.

The principle notion of a projective space is that of **alignment**: that one point x may somehow be "aligned" with two other points a and b. If a permutation which is to preserve alignment leaves the set $\{\mathsf{a},\mathsf{b}\}$ invariant, then it will leave the set M of points aligned with a and b invariant. In particular, if a permutation interchanges a and b, it will do this thing. A curiosity of each projective transformation ϕ which interchanges two points a and b of M is the following: for any x aligned with a and b, there always exists another point y in M distinct from x such that $\{\mathsf{x},\mathsf{y}\} = \{\phi(\mathsf{x}),\phi(\mathsf{y})\}$.

This motivates us to define alignment as follows: an element x is aligned with a and b if, whenever ϕ in $\mathfrak{X}$ interchanges a and b, then there exists y in M distinct from x such that $\{\mathsf{x},\mathsf{y}\} = \{\phi(\mathsf{x}),\phi(\mathsf{y})\}$.

Once "alignment" is made clear, one can move on to the notion of a **fundamental set** F. Fundamental sets are sets on which

(A) a choice of an element of $\mathfrak{X}$ on F can be made freely

and

(B) the values of an element of $\mathfrak{X}$ on the fundamental set determines its values elsewhere as well.

This version differs from the previous one principally in that we emphasize *real* projective spaces and show why these are the most useful in analysis.

It is still a work in progress, as we intend in a subsequent version to show how these ideas can be applied to special relativity.

Section 1. Notation

(1.1) Terminology Most of the terminology employed in the sequel will be of standard set theoretical nature. We shall however introduce some new terms consistent with standard terms, and bring especial specific meaning to a few terms often applied in a more general sense.

Elements of a set will be said to be **contained** by that set, whereas subsets of a set will be said to be **comprehended** by that set.

The symmetric difference of two sets X and Y being denoted by

$$\mathsf{X} \triangle \mathsf{Y} , \tag{1}$$

we note that, if X is a subset of Y, *then* $\mathsf{X} \triangle \mathsf{Y}$ *and* $\mathsf{Y} \triangle \mathsf{X}$ *both represent the set complement of* X *in* Y.

The term **singleton** is commonly attached to a set containing a single element. We shall apply the term **doubleton** to a set containing precisely two elements, the term **tripleton** to a set containing precisely three elements, and so forth.

A **transformation of a set** A **onto a set** B *will mean a one-to-one function of* A *onto* B, A **permutation** is a transformation of one set onto itself.

A **libra of transformations** is a family $\mathcal{L}$ of transformations having a common domain, a common range and such that

$$(\forall \{\alpha,\beta,\gamma\} \subset \mathcal{L}) \quad \alpha \circ \beta^{-1} \circ \gamma \in \mathcal{L} . \tag{2}$$

Thus a **group of transformations** is a libra of transformations for which the common range equals the common domain and which contains the identity permutation ι of that common domain.

If $\mathcal{F}$ is a libra of transformations from one set X onto another Y, if S is a subset of X and if T is a subset of Y, we shall adopt the notation

$$\mathcal{F}\big|_{\mathsf{S},\mathsf{T}} \quad \equiv \quad \{\theta \,|\, \mathsf{S} \hookrightarrow \mathsf{T} : (\exists\, \phi \in \mathcal{F}) \quad \theta \subset \phi\} . \tag{3}$$

We shall say that $\mathcal{F}\big|_{\mathsf{S},\mathsf{T}}$ is the **restriction of the family** $\mathcal{F}$ **to** $\mathsf{S} \times \mathsf{T}$. We note that $\mathcal{F}\big|_{\mathsf{S},\mathsf{T}}$ is also a libra of transformations.

For the case wherein S is identical with T, we shall abbreviate the notation of (3) to

$$\mathcal{F}\big|_{\mathsf{S}} . \tag{4}$$

In this case, if $\mathcal{F}$ is a group, then so is $\mathcal{F}\big|_{\mathsf{S}}$.

If ϕ is any function and X is any subset of the domain of ϕ, we define

$$\phi[\mathsf{X}] \quad \equiv \quad \{\phi(\mathsf{x}) : \mathsf{x} \in \mathsf{X}\} . \tag{5}$$

Section 2. Projective Space

(2.1) Definition We begin with a group

$$\mathcal{P} \tag{1}$$

of permutations on a set

$$\mathsf{P}\,, \tag{2}$$

our object being to prescribe properties for $\mathcal{P}$ which will cause P to resemble the intuitive idea of a projective space.

Perhaps the most basic concept of projective geometry is the idea of alignment: what it means for a point $\mathfrak{p}$ to be aligned with two other points $\mathfrak{a}$ and $\mathfrak{b}$. It is a signal fact that when a projective mapping interchanges two points, it acts as an involution on the set of points aligned with those two points. That means, that if $\mathfrak{p}$ is aligned with $\mathfrak{a}$ and $\mathfrak{b}$, and if ϕ is in $\mathcal{P}$, either $\phi(\mathfrak{p})$ should be $\mathfrak{p}$, or $\phi(\phi(\mathfrak{p}))$ should be $\mathfrak{p}$. It also is a signal fact that, if an involutive projective mapping has a fixed point, it has at least two fixed points. Consequently, if $\mathfrak{p}$ is aligned with points $\mathfrak{a}$ and $\mathfrak{b}$, and if ϕ interchanges $\mathfrak{a}$ and $\mathfrak{b}$, then there always exists a fourth point $\mathfrak{q}$ distinct from $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{p}$ such that $\{\mathfrak{p},\mathfrak{q}\}=\{\phi(\mathfrak{p}),\phi(\mathfrak{q})\}$. This motivates our definition of alignment: a point $\mathfrak{p}$ **is aligned with a doubleton** $\{\mathfrak{a},\mathfrak{b}\}$ if

$$(\forall\,\phi\in\mathcal{P}\colon\ \phi(\mathfrak{a})=\mathfrak{b}\quad\text{and}\quad\phi(\mathfrak{b})=\mathfrak{a})\ (\exists\,\mathfrak{q}\in(\mathsf{P}\triangle\{\mathfrak{p}\}))\qquad\{\mathfrak{p},\mathfrak{q}\}=\{\phi(\mathfrak{p}),\phi(\mathfrak{q})\}\,. \tag{3}$$

The condition $\{\mathfrak{p},\mathfrak{q}\}=\{\phi(\mathfrak{p}),\phi(\mathfrak{q})\}$ for $\mathfrak{q}$, admits of two possibilities: either

$$\phi(\mathfrak{p})=\mathfrak{q}\quad\text{and}\quad\phi(\mathfrak{q})=\mathfrak{p}$$

or

$$\phi(\mathfrak{p})=\mathfrak{p}\quad\text{and}\quad\phi(\mathfrak{q})=\mathfrak{q}\,.$$

For a doubleton $\{\mathfrak{a},\mathfrak{b}\}$ we shall write

$$\overleftrightarrow{\mathfrak{a},\mathfrak{b}} \tag{4}$$

for the set of all points aligned with $\{\mathfrak{a},\mathfrak{b}\}$. Though tempted to call $\overleftrightarrow{\mathfrak{a},\mathfrak{b}}$ a "line", we are dissuaded by the fact that "line" means something else in the context of linear spaces. Instead, we shall call $\overleftrightarrow{\mathfrak{a},\mathfrak{b}}$ a **meridian**.

Obviously both $\mathfrak{a}$ and $\mathfrak{b}$ are aligned with $\{\mathfrak{a},\mathfrak{b}\}$. To provide that our geometry be non-trivial, we posit the following

Postulate of Substance:

$$(\forall\,\{\mathfrak{a},\mathfrak{b}\}\text{ a doubleton subset of }\mathsf{P})\qquad\overleftrightarrow{\mathfrak{a},\mathfrak{b}}\neq\{\mathfrak{a},\mathfrak{b}\}\,. \tag{5}$$

That is, $\overleftrightarrow{\mathfrak{a},\mathfrak{b}}$ **has at least three distinct points**. Of course, if $\overleftrightarrow{\mathfrak{a},\mathfrak{b}}$ has three distinct points, it follows from (3) that it has at least four as well.

If a set has the property that, whenever it comprehends a doubleton $\{\mathfrak{a},\mathfrak{b}\}$, it also comprehends the entire meridian $\overleftrightarrow{\mathfrak{a},\mathfrak{b}}$, we shall say that that set is **extensive**. The set P is extensive, and the intersection of any family of extensive sets is extensive, whence follows that each subset S of P is comprehended by a minimal extensive set. We shall call that set the **span of** S and denote it by

$$\boxed{\mathsf{S}}\,. \tag{6}$$

A subset I of P will be said to be **independent** if there is no element $\mathfrak{p}$ of I for which $\mathfrak{p}$ is in $\boxed{\mathsf{I}\triangle\{\mathfrak{p}\}}$. An element $\mathfrak{x}$ of a subset S of P will be said to be a **fundamental element of** S if it is in $\boxed{\mathsf{S}\triangle\{\mathfrak{x}\}}$ and, for each $\mathfrak{y}$ in $\mathsf{S}\triangle\{\mathfrak{x}\}$, $\{\mathfrak{x},\mathfrak{y}\}\cap\boxed{\mathsf{S}\triangle\{\mathfrak{x},\mathfrak{y}\}}=\varnothing$. If a set has a fundamental element, that set will be said to be a **fundamental set**.

Our second postulate for a projective space shall be the

Uniqueness Postulate:

$$(\forall\,\{\alpha,\beta\}\subset\mathcal{P})(\forall\,\mathsf{F}\text{ a fundamental subset of }\mathsf{P})\quad \alpha\text{ and }\beta\text{ agree on }\boxed{\mathsf{F}}\text{ if they agree on }\mathsf{F}. \tag{7}$$

Our third and final postulate is motivated by the so-called "fundamental theorem of projective geometry":[1]

Fundamental Postulate:

$$(\forall\,\theta\,|\,\mathsf{F}\hookrightarrow\mathsf{G}\colon\ \theta\text{ is a transformation and }\mathsf{F}\text{ and }\mathsf{G}\text{ are fundamental sets})(\exists\,\phi\in\mathcal{P})\quad \theta\subset\phi\,. \tag{8}$$

If all three postulates (5), (7) and (8) are satisfied, we shall say that $\mathcal{P}$ is a **projective permutation group with associated projective space** P. In this case, the elements of $\mathcal{P}$ are said to be **homographies**, and the projective group $\mathcal{P}$ may be referred to as the **homography group of** P.

By a **trivial projective space**, we shall mean a singleton.

In the sequel, P *shall always denote a projective space and* $\mathcal{P}$ *its group of homographies.*

(2.2) Examples A **field** is a set F with two designated elements $\mathtt{o}$ and $\mathtt{l}$ and two binary operations $+$ and $\cdot$ such that $+$ is an abelian group operation on F with identity $\mathtt{o}$, $\cdot$ is an Abelian group operation on $\mathsf{F}\triangle\{\mathtt{o}\}$ with identity $\mathtt{l}$ and such that the "distributive law" holds:

$$(\forall\,\{\mathtt{x},\mathtt{y},\mathtt{x}\}\subset\mathsf{F})\quad \mathtt{x}\cdot(\mathtt{y}+\mathtt{z})\ =\ (\mathtt{x}\cdot\mathtt{y})+(\mathtt{x}\cdot\mathtt{z})\,.$$

If there exists a positive integer $\mathtt{n}$ such that $\mathtt{n}\cdot\mathtt{l}=\mathtt{o}$, then the smallest such integer is said to be the **characteristic of** F.

We form a larger set M by adducing an element ∞ not already an element of F. For elements $\mathtt{a}$, $\mathtt{b}$, $\mathtt{c}$ and $\mathtt{d}$ in F such that $\mathtt{a}\cdot\mathtt{d}\neq\mathtt{b}\cdot\mathtt{c}$, we define the transformation $\left(\begin{smallmatrix}\mathtt{a}&\mathtt{b}\\ \mathtt{c}&\mathtt{d}\end{smallmatrix}\right)|\,\mathsf{M}\hookrightarrow\mathsf{M}$ by, for each $\mathtt{x}$ in M

$$\left(\begin{smallmatrix}\mathtt{a}&\mathtt{b}\\ \mathtt{c}&\mathtt{d}\end{smallmatrix}\right)(\mathtt{x})\quad\equiv\quad\begin{cases}\infty & \text{if } \mathtt{x}=\infty \text{ and } \mathtt{c}=0\,;\\ \frac{\mathtt{a}}{\mathtt{c}} & \text{if } \mathtt{x}=\infty \text{ and } \mathtt{c}\neq 0\,;\\ \infty & \text{if } \mathtt{x}\neq\infty \text{ and } \mathtt{c}\cdot\mathtt{x}=0\,;\\ \frac{\mathtt{a}\cdot\mathtt{x}+\mathtt{b}}{\mathtt{c}\cdot\mathtt{x}+\mathtt{d}} & \text{if } \mathtt{x}\neq\infty \text{ and } \mathtt{c}\cdot\mathtt{x}+\mathtt{d}\neq 0\,.\end{cases} \tag{1}$$

These functions are called **linear fractional transformations** and, if F does not have characteristic 2, they form a projective group on M, where composition corresponds to matrix multiplication.
Examples of fields of characteristic different from 2 can be found by looking at prime numbers $\mathtt{p}$. If we let F be the set $\{0,1,\cdots,\mathtt{p}\text{-}1\}$ and define $+$ and $\cdot$ as addition and multiplication modulo $\mathtt{p}$, then F becomes a field of characteristic $\mathtt{p}$. Hence M as above is a projective space which we shall call

$$\mathbf{M}(\mathtt{p})\,. \tag{2}$$

In the case $\mathtt{p}=3$

$$\text{the projective group is just the family of all permutations of } \mathbf{M}(3) \tag{3}$$

and so has exactly 24 elements.

If F is the field $\mathbb{R}$ of real numbers, the corresponding projective space M is of course infinite. This example, as well as all the examples $\mathbf{M}(\mathtt{p})$ above, has the property that some involutions have fixed points, and some do not. This is the most important example in the realm of analysis, and is special for several reasons. The most important of these reasons is that there is a topology inherent on M induced by the defining projective group. We shall examine that topology in the sequel (3).

[1] We remind the reader that in this account "transformation" means a bijective function.

If F is the field $\mathbb{C}$ of complex numbers, then all involutions have fixed points. There is no inherent topology on F and when this example appears in analysis, one must adopt a topology from context.

Perhaps the most important and intuitive example of a projective space not *a priori* associated with a field is the circle, which we here visualize as a subset $\mathbb{T}$ of the euclidian plane. Each point $\mathfrak{p}$ of the plane not on $\mathbb{T}$ corresponds to an involution $\pi_{\mathfrak{p}}$ of $\mathbb{T}$ which sends each point $\mathfrak{x}$ of $\mathbb{T}$ to the other point $\pi_{\mathfrak{p}}(\mathfrak{x})$ on the intersection of $\mathbb{T}$ with the line passing through both $\mathfrak{x}$ and $\mathfrak{p}$ (or to $\mathfrak{x}$ itself if that line is tangent to $\mathbb{T}$). We shall call these functions $\pi_{\mathfrak{p}}$ **point involutions of** $\mathbb{T}$. If the point $\mathfrak{p}$ is inside the circle $\mathbb{T}$, the corresponding point involution $\pi_{\mathfrak{p}}$ will have no fixed points. If $\mathfrak{p}$ is outside $\mathbb{T}$, then $\pi_{\mathfrak{p}}$ will have two fixed points: the line joining those fixed points will be called the **chord of the involution**. Thus each chord through $\mathbb{T}$ which is not a diameter corresponds to a point involution where the point is outside $\mathbb{T}$. Each diameter chord D however does correspond to its own involution: the reflection π_{D} of $\mathbb{T}$ onto itself through D. Such an involution will be called a **diameter involution**. The permutation group $\mathcal{T}$ acting on $\mathbb{T}$ generated by all these point involutions and diameter involutions can be shown to be a projective group. We shall see later that each element of $\mathcal{T}$ in fact can be written as a composition of two of these involutions.

In all the examples listed thus far, maximal independent sets are doubletons and fundamental sets are tripletons.

(2.3) Projective Isomorphisms Let P be a projective space with projective group $\mathcal{P}$ and let Q be a projective space with projective group $\mathcal{Q}$. A transformation $\phi\,|\,\mathsf{P}\hookrightarrow\mathsf{Q}$ is said to be a **projective isomorphism** if

$$\mathcal{Q} \;=\; \{\phi\circ\theta\circ\phi^{-1}\colon\theta\in\mathcal{P}\}\,. \tag{1}$$

When Q is just P, we call a projective isomorphism a **projective automorphism**. Evidently

$$\text{homographies are projective automorphisms.} \tag{2}$$

The converse of (2) does not always hold. Such is the case, for instance, when the projective space derives from the field of complex numbers: in this case, complex conjugation is a projective automorphism which is not a homography. We take up this idea again in (2.12) *infra*.

(2.4) Initial Observations Let $\{\mathfrak{a},\mathfrak{b}\}$ be a doubleton subset of P. It is trivial that

$$\{\mathfrak{a},\mathfrak{b}\}\subset\overleftrightarrow{\mathfrak{a},\mathfrak{b}}\,. \tag{1}$$

Let ϕ be a projective isomorphism from one projective space P onto another Q. If $\{\mathfrak{a},\mathfrak{b},\mathfrak{x}\}$ is a tripleton such that $\mathfrak{x}$ is aligned with $\{\mathfrak{a},\mathfrak{b}\}$, then

$$(\forall\,\phi\in\mathcal{P})\quad \phi(\mathfrak{x}) \text{ is aligned with } \{\phi(\mathfrak{a}),\phi(\mathfrak{b})\}\,. \tag{2}$$

Let θ be any homography of Q which interchanges $\phi(\mathfrak{a})$ and $\phi(\mathfrak{b})$. Then $\phi^{-1}\circ\theta\circ\phi$ interchanges $\mathfrak{a}$ and $\mathfrak{b}$. Since $\mathfrak{x}$ is aligned with $\{\mathfrak{a},\mathfrak{b}\}$, it follows from (2.1.3) that there exists $\mathfrak{y}\in(\mathsf{P}\triangle\{\mathfrak{a},\mathfrak{b},\mathfrak{x}\})$ such that

$$\{\mathfrak{x},\mathfrak{y}\}=\{\phi^{-1}\circ\theta\circ\phi(\mathfrak{x}),\phi^{-1}\circ\theta\circ\phi(\mathfrak{y})\}\,.$$

Applying ϕ to each side of the above, we obtain

$$\{\phi(\mathfrak{x}),\phi(\mathfrak{y})\}=\{\theta(\phi(\mathfrak{x})),\theta(\phi(\mathfrak{y}))\}\,,$$

whence follows that (2) holds.

Let ϕ be a projective isomorphism from one projective space P onto another Q. Let S be any subset of P. Then

$$\boxed{\phi[\mathsf{S}]}=\phi[\boxed{\mathsf{S}}]\,. \tag{3}$$

Than $S \subset \boxed{S}$ is trivial: it follows that

$$\phi[S] \subset \phi[\boxed{S}]. \tag{i}$$

Suppose that a and b are elements of $\phi[\boxed{S}]$ and that x is aligned with $\{a,b\}$. From (2) we know that $\phi^{-1}(x)$ is aligned with $\phi^{-1}(a)$ and $\phi^{-1}(b)$. Since $\boxed{S}$ is extensive, it follows that $\phi^{-1}(x)$ is in $\boxed{S}$. It follows that x is in $\phi[\boxed{S}]$. We have shown that $\phi[\boxed{S}]$is extensive. It follows from (i) that

$$\boxed{\phi[S]} \subset \phi[\boxed{S}]. \tag{ii}$$

If we replace the function ϕ by the function ϕ^{-1}, and the set S by the set $\phi[S]$ in (i), we obtain from (ii) that $\boxed{S} \subset \phi^{-1}[\boxed{\phi[S]}]$. It follows that

$$\phi[\boxed{S}] \subset \boxed{\phi[S]}. \tag{iii}$$

That (3) holds follows from (ii) and (iii).

Let ϕ be a projective isomorphism from one projective space P onto another Q. Let p be a fundamental point for a fundamental set F. Then

$$\phi[F] \text{ is fundamental and } \phi(p) \text{ is a fundamental point for } \phi[F]. \tag{4}$$

Let b be any element of $(\phi[F])\triangle\{\phi(p)\}$. Then $\phi^{-1}(b)$ is in $F\triangle\{p\}$ and so, since p is fundamental, $\{p,\phi^{-1}(b)\} \cap \boxed{F\triangle\{p,\phi^{-1}(b)\}} = \varnothing$. It follows from (3) that $\phi[\boxed{F\triangle\{p,\phi^{-1}(b)\}}] = \boxed{\phi[F]\triangle\{\phi(p),b\}}$, whence follows that $\phi(p)$ is a fundamental element of $\phi[F]$.

Let $\{a,b,x\}$ be a tripleton comprehended by P. Then

$$x \text{ is aligned with } \{a,b\} \iff x \text{ is a fundamental element of } \{a,b,x\}. \tag{5}$$

$\Longrightarrow$: Since singletons cannot comprehend doubletons, we know that both $\{a\}$ and $\{b\}$ are extensive (by default). It follows that $\{a\} = \boxed{\{a\}}$, that $\{b\} = \boxed{\{b\}}$ and that $\{a,b\}$ is independent. Since x is aligned with $\{a,b\}$, it is in $\boxed{\{a,b\}}$ but in neither $\{a\}$ nor $\{b\}$. It follows that x is fundamental for the set $\{a,b,x\}$.

$\Longleftarrow$: We suppose now that x is fundamental for $\{a,b\}$. It is a consequence of the postulate of substance that there exists an element p distinct from a and b aligned with $\{a,b\}$. It is a consequence of the first paragraph of this proof that p is fundamental for $\{a,b,x\}$. It is a consequence of the fundamental postulate that there exists a homography ϕ which leaves a and b fixed and sends p to x. Since p is aligned with $\{a,b\}$, it follows from (2) that x is also aligned with $\{a,b\}$.

Let F be any fundamental set and let p be any element of F. Then

$$F\triangle\{p\} \text{ is independent and } p \text{ is a fundamental element for } F. \tag{6}$$

Since F is a fundamental set, it has a fundamental element e. From the fundamental postulate we know that there exists an element of $\mathcal{P}$ which interchanges p and e and leaves every other element of F fixed. If p were not a fundamental element of F, then there would be an element t of $F\triangle\{p\}$ such that either p or t were in $\boxed{F\triangle\{t,p\}}$. If t were equal to e, then either p or e would be in $\boxed{F\triangle\{e,p\}}$, which would be absurd since e is a fundamental element of F. Thus t could not be either e or p. By (3), either $\phi(p)$ or $\phi(t)$ would be in $\boxed{\phi[F\triangle\{t,p\}]} = \phi[\boxed{F\triangle\{t,p\}}]$: i.e. either e or t would be in $\boxed{F\triangle\{t,p\}}$. Again this would be impossible since e is a fundamental element of F. It follows that p must be a fundamental element of F.

If $F\triangle\{p\}$ were not independent, then there would exist s in $F\triangle\{p\}$ such that s were in $\boxed{F\triangle\{p,s\}}$. But we know now that p is a fundamental element of F so this could not be possible.

(2.5) Projective Bases A projective space is said to be **finite dimensional** if there is a finite upper bound to the cardinalities of its independent subsets. Let P be a finite dimensional projective space. By an **independent basis for** P we shall mean a maximal independent subset of P. By a **fundamental basis for** P we shall mean a maximal fundamental subset of P. That a fundamental basis exists follows from (2.4.6).

(2.6) Subspace of a Projective Space Let P be a projective space with projective group $\mathcal{P}$ and let S be an extensive subset of P. Such sets S are projective spaces in their own right with projective groups defined by

$$\mathcal{P}_{\mathsf{S}} \quad \equiv \quad \{\phi\big|_{\mathsf{S}} : \phi \in \mathcal{P} \quad \text{and} \quad (\forall\, \mathsf{x} \in \mathsf{S})\ \phi(\mathsf{x}) \in \mathsf{S}\}\,. \tag{1}$$

That $\mathcal{P}_{\mathsf{S}}$ is a projective group follows directly from the postulates and the definitions. We shall at times refer to extensive sets as **projective subspaces**.

(2.7) Homography Notation The fundamental postulate and the uniqueness postulate together suggest introduction of special notation. Let θ be a transformation of one finite fundamental set F onto another G. We know that there exists an element of $\mathcal{P}$ unique on $\boxed{\mathsf{F}}$ which extends θ. If $\mathsf{F} = \{\mathsf{a},\mathsf{b},\mathsf{c},\dots\}$, we shall denote that element of $\mathcal{P}$ by

$$\boxed{\begin{matrix}\theta(\mathsf{a}) & \theta(\mathsf{b}) & \theta(\mathsf{c}) & \cdots \\ \mathsf{a} & \mathsf{b} & \mathsf{c} & \cdots\end{matrix}}\,. \tag{1}$$

For example, the expression

$$\boxed{\begin{matrix}\mathsf{b} & \mathsf{d} & \mathsf{f} \\ \mathsf{a} & \mathsf{c} & \mathsf{e}\end{matrix}} \tag{2}$$

means that $\{\mathsf{a},\mathsf{c},\mathsf{e}\}$ is a tripleton in one meridian comprised by P, that $\{\mathsf{b},\mathsf{d},\mathsf{f}\}$ is another such tripleton and that $\boxed{\begin{matrix}\mathsf{b} & \mathsf{d} & \mathsf{f} \\ \mathsf{a} & \mathsf{c} & \mathsf{e}\end{matrix}}$ is the restriction to $\boxed{\{\mathsf{a},\mathsf{c},\mathsf{e}\}}$ of an homography ϕ of P which sends a to b, c to d and e to f.

For fundamental subsets of meridians, this terminology may be abbreviated, due to the following fact: Let $\{\mathsf{a},\mathsf{b}\}$ be a doubleton in P and let ϕ be any homography which interchanges a and b. Then

$$(\forall\, \mathsf{x} \in \overleftrightarrow{\mathsf{a},\mathsf{b}})\quad \phi^{-1}(\mathsf{x}) = \phi(\mathsf{x})\,. \tag{3}$$

Since x is aligned with $\{\mathsf{a},\mathsf{b}\}$, it follows that there exists y in P such that $\{\mathsf{x},\mathsf{y}\} = \{\phi(\mathsf{x}),\phi(\mathsf{y})\}$. If $\mathsf{x} = \phi(\mathsf{x})$, then $\phi^{-1}(\mathsf{x}) = \phi(\mathsf{x})$ and, if $\mathsf{x} = \phi(\mathsf{y})$, then $\phi^{-1}(\mathsf{x}) = \mathsf{y} = \phi(\mathsf{x})$. In both cases we have (3).

When (3) holds and ϕ does *not* fix each element of $\overleftrightarrow{\mathsf{a},\mathsf{b}}$, we say that ϕ is **involutive on** $\overleftrightarrow{\mathsf{a},\mathsf{b}}$. If for any reason we know that ϕ is involutive, that the terms c and d are distinct and that ϕ is of the form $\boxed{\begin{matrix}\mathsf{b} & \mathsf{d} & \mathsf{f} \\ \mathsf{a} & \mathsf{c} & \mathsf{e}\end{matrix}}$, then ϕ and and $\boxed{\begin{matrix}\mathsf{b} & \mathsf{d} & \mathsf{c} \\ \mathsf{a} & \mathsf{c} & \mathsf{d}\end{matrix}}$ agree on the fundamental set $\{\mathsf{a},\mathsf{b},\mathsf{c}\}$. It follows from the uniqueness postulate that $\boxed{\begin{matrix}\mathsf{b} & \mathsf{d} & \mathsf{f} \\ \mathsf{a} & \mathsf{c} & \mathsf{e}\end{matrix}}$ is the same function as $\boxed{\begin{matrix}\mathsf{b} & \mathsf{d} & \mathsf{c} \\ \mathsf{a} & \mathsf{c} & \mathsf{d}\end{matrix}}$. This suggests that we introduce an abbreviated notation for this expression:

$$\boxed{\begin{matrix}\mathsf{b} & \mathsf{d} \\ \mathsf{a} & \mathsf{c}\end{matrix}}\,. \tag{4}$$

In the sequel, an expression of the form as in (4) will always represent an element of $\mathcal{P}$ which is involutive on the meridian $\overleftrightarrow{\mathsf{a},\mathsf{b}}$, a is sent to b, c is sent to d and $\{\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d}\}$ has cardinality at least 3.

For any projective isomorphism ϕ defined on P, any pair of fundamental subsets $\{\mathsf{a},\mathsf{b},\mathsf{c},\dots\}$ and $\{w,x,y,\dots\}$ of P of equal finite cardinality, it is generally true that

$$(\forall\, \mathsf{t} \in \boxed{\{\mathsf{a},\mathsf{b},\mathsf{c},\dots\}})\qquad \boxed{\begin{matrix}\phi(w) & \phi(x) & \phi(y) & \cdots \\ \phi(\mathsf{a}) & \phi(\mathsf{b}) & \phi(\mathsf{c}) & \cdots\end{matrix}}(\phi(\mathsf{t})) \quad = \quad \phi(\boxed{\begin{matrix}w & x & y & \cdots \\ \mathsf{a} & \mathsf{b} & \mathsf{c} & \cdots\end{matrix}}(\mathsf{t}))\tag{5}$$

Setting $\theta \equiv \phi \circ \begin{bmatrix} w & x & y & \cdots \\ a & b & c & \cdots \end{bmatrix} \circ \phi^{-1}$, we see that θ sends $\phi(a)$ to $\phi(w)$, $\phi(b)$ to $\phi(x)$, $\phi(c)$ to $\phi(y)$ *et cet.* Since θ is a homography, it follows from the (2.4.4) and the uniqueness postulate that $\theta = \begin{bmatrix} \phi(w) & \phi(x) & \phi(y) & \cdots \\ \phi(a) & \phi(b) & \phi(c) & \cdots \end{bmatrix}$. Consequently, for each t in $\{a,b,c,\ldots\}$, we have

$$\phi\left(\begin{bmatrix} w & x & y & \cdots \\ a & b & c & \cdots \end{bmatrix}(t)\right) = \theta(\phi(t)) = \begin{bmatrix} \phi(w) & \phi(x) & \phi(y) & \cdots \\ \phi(a) & \phi(b) & \phi(c) & \cdots \end{bmatrix}(\phi(t)).$$

(2.8) Meridians Let M be a meridian and let $\{p,q\}$ be a doubleton subset of M. Then

$$M = \overleftrightarrow{p,q}. \tag{1}$$

Assume that (1) did not hold. Then there would exist a homography ϕ on M which interchanged p and q and an element c of M such that

$$d \equiv \phi\circ\phi(c) \neq c. \tag{i}$$

It is a consequence of the postulate of substance that there exists an element r of $\overleftrightarrow{p,q}$ distinct from p and q. Since M is a meridian, there exists a doubleton $\{a,b\}$ such that $M = \overleftrightarrow{a,b}$. It follows from (2.4.5) that $\{a,b,c\}$ and $\{p,q,r\}$ would be fundamental sets. It follows from the fundamental postulate that there would exist a homography θ of M such that

$$\theta(a) = p,\ \theta(b) = q \text{ and } \theta(d) = r. \tag{ii}$$

Let

$$\psi \equiv \theta^{-1}\circ\phi\circ\theta \quad \text{and} \quad e \equiv \theta^{-1}(c). \tag{iii}$$

We would have by (i)

$$\psi\circ\psi(e) \overset{\text{by (iii)}}{=\!=} \theta^{-1}\circ\phi\circ\phi(c) \neq \theta^{-1}(c) \overset{\text{by (iii)}}{=\!=} e. \tag{iv}$$

By (ii) and (iv) we would have

$$\psi(a) = b \quad \text{and} \quad \psi(b) = a.$$

But $M = \overleftrightarrow{a,b}$ and e would be in M: thus it would follow from (2.7.3) that $\psi\circ\psi(e) = e$. In view of (iv), that would be absurd.

Let $\{a,b,c\}$ be a tripleton subset of a meridian. Then

$$\{a,b,c\} \text{ is fundamental set.} \tag{2}$$

From (1) we know that $\overleftrightarrow{a,b}$ is just the meridian, and so c is aligned with $\{a,b\}$. From (2.4.5) we know then that $\{a,b,c\}$ is fundamental.

We shall call an involutive homography of a meridian a **meridian involution**. It is an idiosyncrasy that

$$\text{a meridian involution } \phi \text{ either has no fixed point or two fixed points.} \tag{3}$$

If ϕ had three fixed points, we would know from (2) that the set of these points would be a fundamental set and so by the uniqueness postulate ϕ would agree with the identity function on the span of those fixed points. But, by (1), that span would be the whole meridian, and so ϕ could not be a *bona-fide* involution.

Suppose that ϕ has a fixed point x. From the postulate of substance we know that there exists a doubleton $\{a,b\}$ not containing x. Since x is in alignment with $\{a,b\}$, we know that there exists y distinct from x such that $\{x,y\} = \{\phi(x),\phi(y)\}$. Evidently, $\phi(y)$ must be y.

If $\{a,b\}$ is a doubleton subset of a meridian, then

$$\text{there is a unique involution of that meridian which fixes both } a \text{ and } b. \tag{4}$$

We shall denote that involution by

$$\left[\begin{smallmatrix} a & b \\ a & b \end{smallmatrix}\right] \tag{5}$$

From (1) we know that the meridian is just $\overleftrightarrow{a,b}$ and from the postulate of substance we know that there exists some c in $\overleftrightarrow{a,b}$ distinct from a and b. It follows from (3) that there exists an element d of $\overleftrightarrow{a,b}\triangle\{a,b,c\}$ such that

$$\left[\begin{smallmatrix} b & a & c \\ a & b & c \end{smallmatrix}\right](d) = d\,. \tag{i}$$

Direct calculation with (i) shows that

$$\left[\begin{smallmatrix} c & a & d \\ a & c & b \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} b & a & c \\ a & b & c \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} c & a & d \\ a & c & b \end{smallmatrix}\right](a) = a \quad \text{and} \quad \left[\begin{smallmatrix} c & a & d \\ a & c & b \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} b & a & c \\ a & b & c \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} c & a & d \\ a & c & b \end{smallmatrix}\right](b) = b\,. \tag{ii}$$

Since $\left[\begin{smallmatrix} c & a & d \\ a & c & b \end{smallmatrix}\right]$ interchanges a and c, and since $\left[\begin{smallmatrix} b & a & c \\ a & b & c \end{smallmatrix}\right]$ interchanges a and n we know that both $\left[\begin{smallmatrix} c & a & d \\ a & c & b \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} b & a & c \\ a & b & c \end{smallmatrix}\right]$ are involutions. Hence from direct calculation follows that

$$\left[\begin{smallmatrix} c & a & d \\ a & c & b \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} b & a & c \\ a & b & c \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} c & a & d \\ a & c & b \end{smallmatrix}\right] \quad \text{is an involution.} \tag{iii}$$

Let ϕ be any involution of $\overleftrightarrow{a,b}$ which fixes both a and b. Let

$$e \;\equiv\; \phi(c)\,. \tag{iv}$$

Direct calculation shows that $\left[\begin{smallmatrix} c & a & e \\ a & c & b \end{smallmatrix}\right] \circ \phi \circ \left[\begin{smallmatrix} c & a & e \\ a & c & b \end{smallmatrix}\right]$ fixes both c and e and interchanges a and b. It follows from the uniqueness postulate that

$$\left[\begin{smallmatrix} c & a & e \\ a & c & b \end{smallmatrix}\right] \circ \phi \circ \left[\begin{smallmatrix} c & a & e \\ a & c & b \end{smallmatrix}\right] \;=\; \left[\begin{smallmatrix} b & a & c \\ a & b & c \end{smallmatrix}\right]\,. \tag{v}$$

It follows from (3) and (i) that $\left[\begin{smallmatrix} b & a & c \\ a & b & c \end{smallmatrix}\right]$ fixes only c and d. Since $\left[\begin{smallmatrix} c & a & e \\ a & c & b \end{smallmatrix}\right] \circ \phi \circ \left[\begin{smallmatrix} c & a & e \\ a & c & b \end{smallmatrix}\right]$ fixes only c and e, we must have

$$e = d\,. \tag{vi}$$

Since $\left[\begin{smallmatrix} c & a & e \\ a & c & b \end{smallmatrix}\right]$ is an involution, direct calculation shows that

$$\phi \overset{\text{by (v)}}{=\!=\!=} \left[\begin{smallmatrix} c & a & e \\ a & c & b \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} b & a & c \\ a & b & c \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} c & a & e \\ a & c & b \end{smallmatrix}\right] \overset{\text{by (vi)}}{=\!=\!=} \left[\begin{smallmatrix} c & a & d \\ a & c & b \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} b & a & c \\ a & b & c \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} c & a & d \\ a & c & b \end{smallmatrix}\right]\,. \tag{vii}$$

The existence part of (4) is established by (ii) and the uniqueness by (vii).

A homography on a meridian will be said to be a **translation** if it has a single fixed point. If two distinct involutions α and β of a meridian have a common fixed point a, then

$$\alpha\circ\beta \quad \text{is a translation with fixed point } a\,. \tag{6}$$

If $\alpha\circ\beta$ were not a translation, then it would fix another point e distinct from a. Then α and β would agree on the distinct points a and b and so by (5), α and β would be identical: an absurdity. This establishes (6).

If a translation τ fixes a point a, if b is distinct from a, and if c represents $\tau(b)$, then

$$\left[\begin{smallmatrix} a & c \\ a & c \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & c \\ a & b \end{smallmatrix}\right] \;=\; \tau\,. \tag{7}$$

Let a, b and c be as in (7) and let d be $\tau(c)$. Then

$$\tau(a) = a\,. \tag{i}$$

Evidently $\left[\begin{smallmatrix} c & d & b \\ c & b & d \end{smallmatrix}\right] \circ \tau$ interchanges b and c. By (2.7.3),

$$\left[\begin{smallmatrix} c & d & b \\ c & b & d \end{smallmatrix}\right] \circ \tau \quad \text{is an involution.} \tag{ii}$$

We have

$$a \overset{\text{by (ii)}}{=\!=\!=} \left(\left[\begin{smallmatrix} c & d & b \\ c & b & d \end{smallmatrix}\right] \circ \tau\right) \circ \left(\left[\begin{smallmatrix} c & d & b \\ c & b & d \end{smallmatrix}\right] \circ \tau\right)(a) \overset{\text{by (i)}}{=\!=\!=} \left[\begin{smallmatrix} c & d & b \\ c & b & d \end{smallmatrix}\right] \circ \tau \circ \left[\begin{smallmatrix} c & d & b \\ c & b & d \end{smallmatrix}\right](a)\,. \tag{iii}$$

Since $\left[\begin{smallmatrix} c & d & b \\ c & b & d \end{smallmatrix}\right]$ interchanges b and d, it follows from (2.7.3) that

$$\left[\begin{smallmatrix} c & d & b \\ c & b & d \end{smallmatrix}\right] \quad \text{is an involution.} \tag{iv}$$

Thus

$$\begin{smallmatrix}c&d&b\\c&b&d\end{smallmatrix}(a) \overset{\text{by (iii)}}{=\!=\!=} \begin{smallmatrix}c&d&b\\c&b&d\end{smallmatrix}\left(\begin{smallmatrix}c&d&b\\c&b&d\end{smallmatrix}\circ\tau\circ\begin{smallmatrix}c&d&b\\c&b&d\end{smallmatrix}(a)\right) \overset{\text{by (iv)}}{=\!=\!=} \tau\left(\begin{smallmatrix}c&d&b\\c&b&d\end{smallmatrix}(a)\right) . \tag{v}$$

Since τ is a translation, it follows from (i) and (v) that

$$a = \begin{smallmatrix}c&d&b\\c&b&d\end{smallmatrix}(a) . \tag{vi}$$

Thus, besides interchanging b and c, $\begin{smallmatrix}c&d&b\\c&b&d\end{smallmatrix}\circ\tau$ fixes a, whence follows that

$$\begin{smallmatrix}c&d&b\\c&b&d\end{smallmatrix}\circ\tau = \begin{smallmatrix}a&b&c\\a&c&b\end{smallmatrix} . \tag{vii}$$

From (vi) follows that $\begin{smallmatrix}c&d&b\\c&b&d\end{smallmatrix} = \begin{smallmatrix}a&c\\a&c\end{smallmatrix}$ and so (vii) becomes

$$\begin{smallmatrix}a&c\\a&c\end{smallmatrix}\circ\tau = \begin{smallmatrix}a&b&c\\a&c&b\end{smallmatrix}$$

whence follows (7).

Let two translations on a meridian have a common fixed point. Then
if they agree on any other point, they are identical. (8)

Let a be the common fixed point and let b be the point distinct from a at which they agree, and let c be their common value at b. Then it follows from (7) that both equal $\begin{smallmatrix}a&c\\a&c\end{smallmatrix}\circ\begin{smallmatrix}a&c\\a&b\end{smallmatrix}$.

Let α, β and γ be involutions of a meridian with a common fixed point. Then

$$\alpha\circ\beta\circ\gamma \text{ is an involution.} \tag{9}$$

Let a denote the common fixed point and let x be any other point distinct from a. Let y be $\alpha\circ\beta(x)$ and let z be $\gamma(x)$. It follows from (6) that both $\alpha\circ\beta$ and $\begin{smallmatrix}a&z\\a&y\end{smallmatrix}\circ\gamma$ are translations. Both these translations send a to a and x to y. It follows from (8) that they are identical. Consequently $\alpha\circ\beta\circ\gamma = \begin{smallmatrix}a&z\\a&y\end{smallmatrix}$.

Each fundamental subset $\{\infty,o,1\}$ of a meridian M induces a field F of characteristic not equal to 2, which we shall call the **projective field of P with ordered fundamental set** $[\infty,o,1]$. The definitions are as follows: $F \equiv M\triangle\{\infty\}$, $G \equiv F\triangle\{o\}$,

$$\begin{gathered}(\forall\,\{x,y\}\subset F)\quad x+y \equiv \begin{smallmatrix}\infty&x\\\infty&y\end{smallmatrix}(o)\ ,\quad x+(-x)=o\ ,\\ (\forall\,\{x,y\}\subset G)\quad x\cdot y \equiv \begin{smallmatrix}\infty&x\\o&y\end{smallmatrix}(1)\ ,\quad x\cdot o \equiv o\cdot x \equiv o\ ,\quad x\cdot(x^{-1})=1 \quad\text{and}\quad \frac{x}{y}=x\cdot(y^{-1})\ .\end{gathered} \tag{10}$$

We first note that the involutions $\begin{smallmatrix}\infty&x+y\\\infty&o\end{smallmatrix}$ and $\begin{smallmatrix}\infty&y\\\infty&x\end{smallmatrix}$ agree at both ∞ and o, so

$$\begin{smallmatrix}\infty&x+y\\\infty&o\end{smallmatrix} = \begin{smallmatrix}\infty&y\\\infty&x\end{smallmatrix} . \tag{i}$$

It follows from (9) that $\begin{smallmatrix}\infty&o\\\infty&x\end{smallmatrix}\circ\begin{smallmatrix}\infty&o\\\infty&o\end{smallmatrix}\circ\begin{smallmatrix}\infty&o\\\infty&y\end{smallmatrix}$ is an involution — consequently it is the same as $\begin{smallmatrix}\infty&o\\\infty&y\end{smallmatrix}\circ\begin{smallmatrix}\infty&o\\\infty&o\end{smallmatrix}\circ\begin{smallmatrix}\infty&o\\\infty&x\end{smallmatrix}$. Direct computation shows that the latter agrees with the involution $\begin{smallmatrix}\infty&y\\\infty&x\end{smallmatrix}$ at both ∞ and x. So we have

$$\begin{smallmatrix}\infty&o\\\infty&x\end{smallmatrix}\circ\begin{smallmatrix}\infty&o\\\infty&o\end{smallmatrix}\circ\begin{smallmatrix}\infty&o\\\infty&y\end{smallmatrix} = \begin{smallmatrix}\infty&y\\\infty&x\end{smallmatrix} . \tag{ii}$$

Similar reasoning shows that, for $\{x,y\}$ disjoint from $\{\infty,o\}$,

$$\begin{smallmatrix}\infty&x\cdot y\\o&1\end{smallmatrix} = \begin{smallmatrix}\infty&y\\o&x\end{smallmatrix} = \begin{smallmatrix}\infty&x\\o&1\end{smallmatrix}\circ\begin{smallmatrix}\infty&1\\o&1\end{smallmatrix}\circ\begin{smallmatrix}\infty&y\\o&1\end{smallmatrix} . \tag{iii}$$

For x in F and y in $F\triangle\{o\}$ we define

$$-x \equiv \begin{smallmatrix}\infty&o\\\infty&o\end{smallmatrix}(x) \quad\text{and}\quad u^{-1} \equiv \begin{smallmatrix}\infty&1\\o&1\end{smallmatrix}(u) . \tag{iv}$$

Then

$$x+o \overset{\text{by (i)}}{=\!=\!=} \begin{smallmatrix}\infty&o\\\infty&x\end{smallmatrix}(o)=x\ ,\quad u\cdot 1 \overset{\text{by (iii)}}{=\!=\!=} \begin{smallmatrix}\infty&u\\o&1\end{smallmatrix}(1)=u\ , \tag{v}$$

$$x+(-x) \overset{\text{by (i)}}{=\!=\!=} \begin{smallmatrix}\infty&x\\\infty&-x\end{smallmatrix}(o) \overset{\text{by (ii)}}{=\!=\!=} \begin{smallmatrix}\infty&o\\\infty&-x\end{smallmatrix}\circ\begin{smallmatrix}\infty&o\\\infty&o\end{smallmatrix}\circ\begin{smallmatrix}\infty&x\\\infty&o\end{smallmatrix}(o) = \begin{smallmatrix}\infty&o\\\infty&-x\end{smallmatrix}\circ\begin{smallmatrix}\infty&o\\\infty&o\end{smallmatrix}(x) \overset{\text{by (iv)}}{=\!=\!=} \begin{smallmatrix}\infty&o\\\infty&-x\end{smallmatrix}(-x)=o \tag{vi}$$

and, similarly, one obtains

$$u^{-1}\cdot u = 1\ . \tag{vii}$$

For $\{x,y,z\}$ comprised by F,

$$x+(y+z) \overset{\text{by (i)}}{=\!=} \left[\begin{smallmatrix}\infty & y+z\\ \infty & x\end{smallmatrix}\right](o) \overset{\text{by (ii)}}{=\!=} \left[\begin{smallmatrix}\infty & o\\ \infty & x\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & o\\ \infty & o\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & o\\ \infty & y+z\end{smallmatrix}\right](o) \overset{\text{by (ii)}}{=\!=}$$

$$\left[\begin{smallmatrix}\infty & o\\ \infty & x\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & o\\ \infty & o\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & o\\ \infty & y\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & o\\ \infty & o\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & o\\ \infty & z\end{smallmatrix}\right](o) \overset{\text{by (ii)}}{=\!=} \tag{viii}$$

$$\left[\begin{smallmatrix}\infty & o\\ \infty & x+y\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & o\\ \infty & o\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & o\\ \infty & z\end{smallmatrix}\right](o) \overset{\text{by (ii)}}{=\!=} \left[\begin{smallmatrix}\infty & x+y\\ \infty & z\end{smallmatrix}\right](o) = (x+y)+z\,.$$

One can deduce in a similar way that, for $\{u,v,w\}\subset(F\triangle\{o\})$,

$$u\cdot(v\cdot w) = (u\cdot v)\cdot w\,. \tag{ix}$$

That

$$x+y = y+x \quad \text{and} \quad x\cdot y = y\cdot x \tag{x}$$

is trivial. We have established that "+" and "·" are abelian group operations. It remains to show that the "distributive law" holds.

Let x, y and z be elements of F such that x and $y+z$ are distinct from o. We define

$$\theta \;\equiv\; \left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & y\\ \infty & z\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right]\,. \tag{xi}$$

Evidently

$$\theta(\infty) \;=\; \infty\,. \tag{xii}$$

Furthermore

$$\theta(o) \overset{\text{by (xi)}}{=\!=} \left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & y\\ \infty & z\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right](o) = \left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & y\\ \infty & z\end{smallmatrix}\right](o) \overset{\text{by (i)}}{=\!=}$$

$$\left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right](y+z) = \left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & y+z\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & y+z\end{smallmatrix}\right](y+z) = \tag{xiii}$$

$$\left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & y+z\end{smallmatrix}\right](l) \overset{\text{by (iii)}}{=\!=} x\cdot(y+z)\,.$$

Moreover

$$\theta(x\cdot y) \overset{\text{by (xi)}}{=\!=} \left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & y\\ \infty & z\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right](x\cdot y) \overset{\text{by (iii)}}{=\!=} \left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & y\\ \infty & z\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & x\\ o & y\end{smallmatrix}\right](l) =$$

$$\left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & y\\ \infty & z\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & x\\ o & y\end{smallmatrix}\right]\circ\left(\left[\begin{smallmatrix}\infty & x\\ o & y\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & x\\ o & y\end{smallmatrix}\right]\right)(l) \overset{\text{by (iii)}}{=\!=}$$

$$\left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & y\\ \infty & z\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & x\\ o & y\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & x\\ o & y\end{smallmatrix}\right]\circ\left(\left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & y\\ o & l\end{smallmatrix}\right]\right)(l) =$$

$$\left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & y\\ \infty & z\end{smallmatrix}\right]\circ\left(\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right]\circ\left(\left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right]\circ\left(\left[\begin{smallmatrix}\infty & x\\ o & y\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & x\\ o & y\end{smallmatrix}\right]\right)\circ\left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right]\right)\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right]\right)\circ\left[\begin{smallmatrix}\infty & y\\ o & l\end{smallmatrix}\right](l) =$$

$$\left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & y\\ \infty & z\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & y\\ o & l\end{smallmatrix}\right](l) = \left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right](z) = \left[\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}\right]\circ\left[\begin{smallmatrix}\infty & z\\ o & l\end{smallmatrix}\right](l) \overset{\text{by (iii)}}{=\!=} x\cdot z\,.$$

Since θ is an involution, it follows from this last and (xii) that $\theta = \left[\begin{smallmatrix}\infty & x\cdot z\\ \infty & x\cdot y\end{smallmatrix}\right]$. Consequently

$$\theta(o) = \left[\begin{smallmatrix}\infty & x\cdot z\\ \infty & x\cdot y\end{smallmatrix}\right](o) \overset{\text{by (i)}}{=\!=} x\cdot y+x\cdot z\,. \tag{xiv}$$

From (xiii) and (xiv) we have

$$x\cdot(y+z) = x\cdot y+x\cdot z\,. \tag{xv}$$

We have established that F is a field, if its characteristic were 2, then, for any x in$F\triangle\{0\}$,

$$\left[\begin{smallmatrix}\infty & x\\ \infty & x\end{smallmatrix}\right](o) = x+x = o = o+o = \left[\begin{smallmatrix}\infty & o\\ \infty & o\end{smallmatrix}\right](o)\,.$$

Thus, $\left[\begin{smallmatrix}\infty & x\\ \infty & x\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\infty & o\\ \infty & o\end{smallmatrix}\right](o)$ would agree on two distinct points and, being involutions, would have to be identical. It would follow that $\left[\begin{smallmatrix}\infty & o\\ \infty & o\end{smallmatrix}\right]$ would fix every point in F. Since the identity function is not an involution, this were absurd.

(2.9) Decomposition Let ϕ be a non-involutive homography of a meridian M. Then

$$\text{there exist two involutions } \alpha \text{ and } \beta \text{ of } M \text{ such that } \phi = \alpha\circ\beta\,. \tag{1}$$

Since the composition of any involution with itself is the identity, we may presume that ϕ is not the identity. Since ϕ is not an involution, there exists a point p in M such that $\phi\circ\phi(p)\neq p$. It follows from (2.7.3) that $\left[\begin{smallmatrix}p & \phi(p) & \phi\circ\phi(p)\\ \phi\circ\phi(p) & \phi(p) & p\end{smallmatrix}\right]$ is an involution. Direct computation shows that $\left[\begin{smallmatrix}p & \phi(p) & \phi\circ\phi(p)\\ \phi\circ\phi(p) & \phi(p) & p\end{smallmatrix}\right]\circ\phi$ interchanges p and $\phi(p)$, and so, by (2.7.3), $\left[\begin{smallmatrix}p & \phi(p) & \phi\circ\phi(p)\\ \phi\circ\phi(p) & \phi(p) & p\end{smallmatrix}\right]\circ\phi$ is an involution. Consequently, ϕ is a composition of two involutions:

$$\phi \;=\; \begin{bmatrix} p & \phi(p) & \phi\circ\phi(p) \\ \phi\circ\phi(p) & \phi(p) & p \end{bmatrix} \circ \left(\begin{bmatrix} p & \phi(p) & \phi\circ\phi(p) \\ \phi\circ\phi(p) & \phi(p) & p \end{bmatrix} \circ \phi \right).$$

(2.10) Fundamental Structure of a Projective Space We recall that by an independent basis for an extensive subset S of P we mean a maximal independent subset of S. The least upper bound of the cardinalities of independent bases, minus 1, is the **dimension of** S. If this is finite, we say that S is **finite dimensional**. Towards constructing bases, we have the following

$$(\forall\, I \text{ independent and finite})(\forall\, x \in (P \triangle \boxed{I}))\quad \{x\} \cup I \text{ is an independent basis for } \boxed{\{x\} \cup I}. \tag{1}$$

We establish (1) by induction on the cardinality n of I. It is evidently true for $n = 1$. We suppose that it is known to be true for all n less than or equal to some fixed positive integer m. Assume that (1) were not true for some independent set I of cardinality $m+1$ and some x in $P \triangle \boxed{I}$. Then there would exist some p in I also contained in $\boxed{\{x\} \cup (I \triangle \{p\})}$. If F were any finite proper subset of $\{x\} \cup (I \triangle \{p\})$ such that p were in $\boxed{F}$, then x would have to be in F since I is independent. By our induction hypothesis we would know that F were independent. Since F would be independent and its cardinality would be m or less, this would contradict our induction hypothesis, so such an F could not exist. It follows from the induction hypothesis that

$$\{x\} \cup (I \triangle \{p\}) \text{ would be independent.} \tag{i}$$

Consequently, p would be a fundamental point for $\{x\} \cup (I \triangle \{p\})$. It would follow from the fundamental postulate that there would exist a homography which interchanged x and p but kept each other element of I fixed. It would then follow from (i) and (2.4.4) that $\{x\} \cup I$ would be independent, which would vitiate our assumption.

If P is finite dimensional and d is its dimension, then any independent subset of P of cardinality $d+1$ is said to be a **simplex for** P. It is a consequence of (1) that

$$\text{if } P \text{ is finite dimensional, then any simplex for } P \text{ is an independent basis for } P. \tag{2}$$

Let I be a finite independent set and let p be a fundamental point for $\{p\} \cup I$. Then

$$(\forall\, q \in I)\quad \{p\} \cup (I \triangle \{q\}) \text{ is independent and } q \text{ is fundamental for } \{p\} \cup I. \tag{3}$$

Since $\{p\} \cup I$ is a fundamental set, it follows from them fundamental postulate that there exists an homography which interchanges $\{p\}$ with $\{q\}$ and leaves the other elements of I fixed. It follows from (2.4.4) that $\{q\}$ is a fundamental point for $\boxed{\{p\} \cup (I \triangle \{q\})}$.

If P is of finite dimension d and

$$\text{if } S \text{ is a set of cardinality } d+1 \text{ or less such that } P = \boxed{S}, \text{ then } S \text{ is independent.} \tag{4}$$

Let S be a minimal set which spans P. If S were not independent, then there would be an element p of S in $\boxed{S \triangle \{p\}}$. Then P would equal $\boxed{S \triangle \{p\}}$, which were absurd.

For any independent set I, we shall write

$$\overset{\square}{I} \;\equiv\; \{x \in P : x \text{ is fundamental for the set } \{x\} \cup I.\} \tag{5}$$

Then

$$(\forall\, I \subset P \text{ independent})\quad \{\overset{\square}{J} : J \subset I\} \text{ is a partition of } \boxed{I}. \tag{6}$$

Let x be an element of $\boxed{I}$. Then there is a minimal subset S of I such that X is in $\boxed{S}$. The minimality of S insures that p is fundamental for S. Thus $\boxed{I} \subset \{\overset{\square}{J} : J \subset I\}$.

Now, let S and T be any two distinct subsets of I, and assume that there existed an element p of $\overset{\square}{S} \cap \overset{\square}{T}$. Since S and T are distinct, at least one of them has a point s which is not in the other. Without loss of generality we may and shall suppose that it is S. It follows from (1) that $\{p\} \cup (S \triangle \{s\})$ were independent and that

$$\boxed{S} \;=\; \boxed{\{p\} \cup (S \triangle \{s\})}\,. \tag{i}$$

If T were comprehended by S, then $T \subset (S \triangle \{s\})$ and so $\{p\} \cup T$ would be independent, which is impossible due to p having to be in $\overset{\square}{T}$. We shall enumerate $T \triangle S$ by $\{u_1, \ldots, u_m\}$. Because $S \cup (T \triangle S) = (S \cup T) \subset I$, we know that $S \cup (T \triangle S)$ is independent. Since u_1 is not in $\boxed{S}$, it could not be in $\boxed{\{p\} \cup (S \triangle \{s\})}$ and so by (1) it follows that

$$\boxed{\{u_1\} \cup \{p\} \cup (S \triangle \{s\})} \;=\; \boxed{\{u_1\} \cup \boxed{\{p\} \cup (S \triangle \{s\})}} \;\overset{\text{by (i)}}{=\!=}\; \boxed{\{u_1\} \cup \boxed{S}} \;=\; \boxed{\{u_1\} \cup S}$$

and that $\{u_1, p\} \cup (S \triangle \{s\})$ would be independent. If we apply this same argument $m-1$ more times, we obtain

$$\boxed{\{u_1, \ldots, u_m\} \cup \{p\} \cup (S \triangle \{s\})} \;=\; \boxed{\{u_1, \ldots, u_m\} \cup S} \;=\; \boxed{S \cup T}$$

and that $\{u_1, \ldots, u_m, p\} \cup (S \triangle \{s\})$ would be independent. But $T \subset \{u_1, \ldots, u_m, p\} \cup (S \triangle \{s\})$ and $p \in \overset{\square}{T}$, and so $\{u_1, \ldots, u_m, p\} \cup (S \triangle \{s\})$ could not be independent. It follows that our assumption is absurd, which implies that $S \cap T = \varnothing$.

For x in $\overset{\square}{J}$ with $J \subset I$ we shall say that J is the **frame of** x **relative to** I.

Let I be a finite independent set, let p be an element of I, let q be an element of $\overline{\overline{I \triangle \{p\}}}$ and let x be an element of $\overleftrightarrow{p,q} \triangle \{p,q\}$ (as provided by the postulate of substance). Then

$$x \text{ is in } \overset{\square}{I}. \tag{7}$$

We shall show (7) by induction on the cardinality n of I. It is evident that (7) holds when n equals 2. So we shall suppose that we have established it for all n less than or equal to some m greater than or equal to 2. We then look at I of cardinality equal to $m+1$. That x is in $\boxed{I}$ is evident. Assume that x were not fundamental for I. It follows from (6) that there were a proper subset J of I such that x were in $\overset{\square}{J}$. If p were not in J, then $\overleftrightarrow{x,q}$ would be comprehended by $\boxed{P \triangle \{p\}}$ and, since p is in $\overleftrightarrow{x,p}$, this would imply that p were in J: an absurdity. Thus p would have to be in J. Since q is in the line $\overleftrightarrow{p,x}$, it would follow from our induction hypothesis that q would be in $\overset{\square}{J}$. But J and $(P \triangle \{p\})$ would be distinct subsets of I and so by (2) could not have a common point q. It follows that (7) holds.

Let S be a finite dimensional subspace of P, let T be a maximal proper subspace of S and let M be a meridian comprehended by S but not by T. Then

$$M \text{ intersects } T \text{ at precisely one point.} \tag{8}$$

If M intersected T at two points a and b, by (1), M would be $\overleftrightarrow{a,b}$ and so a subset of T : thus the intersection is either void or a singleton. Let p be any element of M. Since T is a maximal proper subspace of S, we know that $\boxed{\{p\} \cup T} = S$. Let q be a point of M distinct from p. Let I be an independent set which spans T. From (6) we know that exists a subset J of I such that q is in $\overset{\square}{J}$. If p were not in J, then q

would be in a subset of T and (8) would be true, so we shall presume that p is in J. It follows from (7) that their exists an element r of J such that the line $\overleftrightarrow{\mathsf{p},\mathsf{r}}$ intersects $\overline{\mathsf{J}\triangle\{\mathsf{p}\}}$. Since $(\{\mathsf{q}\}\cup\mathsf{J})$ and $(\{\mathsf{r}\}\cup\mathsf{J})$ are both fundamental, it follows from the fundamental postulate that there exists ϕ in $\mathcal{P}$ which interchanges q and r and leaves each element of J fixed. Recall that the meridian $\overleftrightarrow{\mathsf{p},\mathsf{r}}$ intersects $\overline{\mathsf{J}\triangle\{\mathsf{p}\}}$ at some point y. It follows its image by ϕ, which is $\overleftrightarrow{\mathsf{p},\mathsf{q}}$, intersects $\boxed{\mathsf{J}\triangle\{\mathsf{p}\}}$, which is comprehended by T. Thus $\overleftrightarrow{\mathsf{p},\mathsf{q}}$ intersects T at $\phi(\mathsf{y})$.

(2.11) The von Staudt Meridian There is a special meridian inherent in any finite dimensional projective space P. To realize this, we first take any set of cardinality 4: say

$$\mathbf{III}_4 \;\equiv\; \{\heartsuit,\spadesuit,\clubsuit,\diamondsuit\}\,. \tag{1}$$

Then we consider the set

$$\Omega_{\mathsf{P}} \tag{2}$$

consisting of all functions $\mathtt{f}\,\big|\,\mathbf{III}_4 \hookrightarrow \mathsf{P}$ such that the set $\{\mathtt{f}(\mathrm{x}) : \mathrm{x}\in\mathbf{III}_4\}$ is co-aligned and of cardinality at least 3. For such a co-aligned[2] set $\{w,x,y,z\}$ within P, we shall adopt the notation

$$\boxed{\begin{matrix} w & x & y & z \\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit \end{matrix}} \tag{3}$$

for the function on Ω_{P} which sends $\heartsuit$ to w, $\spadesuit$ to x, $\clubsuit$ to y and $\diamondsuit$ to z. We shall adopt the notation

$$\boxed{\begin{matrix} \heartsuit & \spadesuit & \clubsuit & \diamondsuit \\ w & x & y & z \end{matrix}} \tag{4}$$

for the inverse of $\boxed{\begin{matrix} w & x & y & z \\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit \end{matrix}}$.

Besides the identity permutation of $\mathbf{III}_4$, there are three other permutations of particular interest here: the one that interchanges $\heartsuit$ with $\diamondsuit$ and $\spadesuit$ with $\clubsuit$, the one that interchanges $\spadesuit$ with $\diamondsuit$ and $\clubsuit$ with $\heartsuit$, and the one that interchanges $\clubsuit$ with $\diamondsuit$ and $\spadesuit$ with $\heartsuit$: we shall denote them, respectively, by

$$\boxed{\begin{matrix} \heartsuit & \spadesuit \\ \diamondsuit & \clubsuit \end{matrix}}\,,\quad \boxed{\begin{matrix} \spadesuit & \clubsuit \\ \diamondsuit & \heartsuit \end{matrix}}\,,\quad \text{and}\quad \boxed{\begin{matrix} \clubsuit & \spadesuit \\ \diamondsuit & \heartsuit \end{matrix}}\,, \tag{5}$$

respectively. We shall denote that group of four permutations by[3]

$$K. \tag{6}$$

We define, for all sets $\{\mathtt{f},\mathtt{g}\}$ comprised by Ω_{P},

$$\mathtt{f} \overset{\mathsf{P}}{\approx} \mathtt{g} \qquad\Longleftrightarrow\qquad (\exists\,\phi\in\mathcal{P} \text{ and } \kappa\in K) \quad \phi\circ\mathtt{f}\circ\kappa \;=\; \mathtt{g}\,. \tag{7}$$

We shall denote the family of $\overset{\mathsf{P}}{\approx}$-equivalence classes by

$$\mathfrak{W}_{\mathsf{P}}\,. \tag{8}$$

Von Staudt used the terminology "Wurf" for an element of $\mathfrak{W}_{\mathsf{P}}$: we shall use here the English translation **throw**. For any element $\mathtt{t}$ of Ω_{P}, we shall denote the $\overset{\mathsf{P}}{\approx}$-equivalence class or throw containing it by

$$\boxed{\mathtt{t}}\,. \tag{9}$$

Members of Ω_{P} with ranges of cardinality less than 4 all fall inside the three distinct equivalence classes:

[2] By **co-aligned** we mean that each point is aligned with each doubleton in the set: all the elements of the set are in a single meridian.

[3] Sometimes called the **Klein 4-group**.

$$\boldsymbol{\infty} \equiv \{\boxed{\begin{smallmatrix}a&b&c&a\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit\end{smallmatrix}} : \{a,b,c\} \text{ is coaligned}\} \cup \{\boxed{\begin{smallmatrix}a&b&b&c\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit\end{smallmatrix}} : \{a,b,c\} \text{ is coaligned}\}, \tag{10}$$

$$\mathbf{0} \equiv \{\boxed{\begin{smallmatrix}a&b&c&b\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit\end{smallmatrix}} : \{a,b,c\} \text{ is coaligned}\} \cup \{\boxed{\begin{smallmatrix}a&b&a&c\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit\end{smallmatrix}} : \{a,b,c\} \text{ is coaligned}\} \tag{11}$$

and

$$\mathbf{1} \equiv \{\boxed{\begin{smallmatrix}a&b&c&c\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit\end{smallmatrix}} : \{a,b,c\} \text{ is coaligned}\} \cup \{\boxed{\begin{smallmatrix}a&a&b&c\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit\end{smallmatrix}} : \{a,b,c\} \text{ is coaligned}\}. \tag{12}$$

Let M be any meridian comprised by P and let $\{a,b,c\}$ be any tripleton comprised by M. We shall call the function

$$\phi_{a,b,c}\big|M \ni x \hookrightarrow \boxed{\begin{smallmatrix}a&b&c&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit\end{smallmatrix}} \in \mathfrak{W}_P \tag{13}$$

a **von Staudt scalar function on** M. The family

$$\mathcal{W}_P \equiv \{\alpha\circ\beta^{-1}\colon \alpha \text{ and } \beta \text{ are von Staudt scalar functions on } M\} \tag{14}$$

is a projective group of permutations of $\mathfrak{W}_P$, relative to which $\mathfrak{W}_P$ is a meridian. Furthermore $\mathcal{W}_P$ does not depend on the choice of M. For each tripleton $\{u,v,w\}$ in M and each tripleton $\{\mathbf{r},\mathbf{s},\mathbf{t}\}$ of throws, there is a unique von Staudt scalar function on M which sends u to $\mathbf{r}$, v to $\mathbf{s}$ and w to $\mathbf{t}$: we shall denote it by

$$\boxed{\begin{smallmatrix}\mathbf{r}&\mathbf{s}&\mathbf{t}\\ u&v&w\end{smallmatrix}} \quad \text{and its inverse by} \quad \boxed{\begin{smallmatrix}u&v&w\\ \mathbf{r}&\mathbf{s}&\mathbf{t}\end{smallmatrix}}. \tag{15}$$

The inverses of von Staudt scalar functions will be called **von Staudt injections**. A direct calculation shows that, for the function of (13),

$$\phi_{a,b,c} = \boxed{\begin{smallmatrix}\boldsymbol{\infty}&\mathbf{0}&\mathbf{1}\\ a&b&c\end{smallmatrix}}. \tag{16}$$

We shall call $\mathfrak{W}_P$ the **von Staudt meridian for** P, and we shall say that $\{\boldsymbol{\infty},\mathbf{0},\mathbf{1}\}$ is the **canonical fundamental subset of** $\mathfrak{W}_P$. We shall denote the projective field of $\mathfrak{W}_P$ with ordered fundamental set $[\boldsymbol{\infty},\mathbf{0},\mathbf{1}]$ by

$$\mathfrak{F}_P \tag{17}$$

and we shall call it the **von Staudt field of** P.

Let M be a meridian comprehended by P and let $\{\infty,o,l\}$ be a tripleton subset of M. Let F denote the projective field of M with ordered fundamental set $[\infty,o,l]$. Perhaps the most easy to compute of the functions of (15) is $\boxed{\begin{smallmatrix}\boldsymbol{\infty}&\mathbf{0}&\mathbf{1}\\ \infty&o&l\end{smallmatrix}}$: we have

$$\boxed{\begin{smallmatrix}\boldsymbol{\infty}&\mathbf{0}&\mathbf{1}\\ \infty&o&l\end{smallmatrix}}\big|M \ni x \hookrightarrow \boxed{\begin{smallmatrix}\infty&o&l&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit\end{smallmatrix}} \in \mathfrak{W}_P. \tag{18}$$

With the aid of (18) it is not difficult to show that

$$\boxed{\begin{smallmatrix}\boldsymbol{\infty}&\mathbf{0}&\mathbf{1}\\ \infty&o&l\end{smallmatrix}}\Big|_F \quad \text{and} \quad \boxed{\begin{smallmatrix}\infty&o&l\\ \boldsymbol{\infty}&\mathbf{0}&\mathbf{1}\end{smallmatrix}}\Big|_{\mathfrak{F}} \quad \text{are field isomorphisms.} \tag{19}$$

Now suppose the P is finite dimensional and that O is a maximal proper subspace of P. We shall write A for the set $P\triangle O$. Let o be any element of A. Relative to O and o we shall define the **von Stuadt scalar multiplication** of a throw $\mathbf{s}$ times an element x of $A\triangle\{o\}$. By (2.10.8) there exists a unique element ∞ of H which is aligned with o and x. We define

$$\mathbf{s}\bullet x = \boxed{\begin{smallmatrix}\infty&o&x\\ \boldsymbol{\infty}&\mathbf{0}&\mathbf{1}\end{smallmatrix}}(\mathbf{s}). \tag{20}$$

If l is any element of $\overleftrightarrow{o,x}\triangle\{\infty\}$, then there exists precisely one element s of M such that

$$\boxed{\begin{matrix}\infty & o & 1 & s\\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit\end{matrix}} = \mathbf{s}. \tag{21}$$

Then

$$\mathbf{s}\bullet\mathbf{x} \quad \equiv \quad \mathbf{s}\cdot\mathbf{x} \tag{22}$$

where the product $\mathbf{s}\cdot\mathbf{x}$ is relative to the field associated with $\{\infty,o,1\}$.

We have

$$\mathbf{s}\bullet\mathbf{x} \overset{\text{by (20)}}{=\!=} \boxed{\begin{matrix}\infty & o & x\\ \infty & o & 1\end{matrix}}(\mathbf{s}) = \boxed{\begin{matrix}\infty & o & x\\ \infty & o & 1\end{matrix}} \circ \boxed{\begin{matrix}\infty & o & 1\\ \infty & o & 1\end{matrix}}(\mathbf{s}) \overset{\text{by (21)}}{=\!=} \boxed{\begin{matrix}\infty & o & x\\ \infty & o & 1\end{matrix}} \circ \boxed{\begin{matrix}\infty & o & 1\\ \infty & o & 1\end{matrix}} \left(\boxed{\begin{matrix}\infty & o & 1 & s\\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit\end{matrix}} \right) \overset{\text{by (18)}}{=\!=}$$

$$\boxed{\begin{matrix}\infty & o & x\\ \infty & o & 1\end{matrix}}(\mathbf{s}) = \boxed{\begin{matrix}\infty & x\\ o & s\end{matrix}} \circ \boxed{\begin{matrix}\infty & s\\ o & 1\end{matrix}}(\mathbf{s}) = \boxed{\begin{matrix}\infty & x\\ o & s\end{matrix}}(1) \overset{\text{by (2.8.10)}}{=\!=} \mathbf{s}\cdot\mathbf{x}.$$

(2.12) Isographies We shall say that a projective isomorphism θ from one projective space P onto itself is **autographic** if

$$(\forall\, a \in \Omega_{\mathsf{P}}) \quad a \overset{\mathsf{P}}{\approx} \theta\circ a\,, \tag{1}$$

where $\overset{\mathsf{P}}{\approx}$ is the equivalence relation associated with the von Staudt meridian of P. If P is finite dimensional, then

a projective automorphism θ is a homography if, and only if, it is autographic. (2)

That a homography is autographic is trivial.

Let θ be autographic and let F be a maximal fundamental subset of P. That $\theta[\mathsf{F}]$ is a fundamental set follows from (2.4.4).It follows from the fundamental postulate that there exists an element ψ of $\mathcal{P}$ which agrees with θ on F. Let $\mathtt{o}$ be any element of F and let I be $\mathsf{F}\triangle\{\mathtt{o}\}$. Thus I is an independent basis for P. We shall first show that

$$(\forall\, \mathsf{J}\subset\mathsf{I})(\exists\, \mathtt{j}\in\boxed{\mathsf{J}}) \quad \theta(\mathtt{j}) = \psi(\mathtt{j})\,. \tag{i}$$

(i) is evidently true for the case $\mathsf{J}\equiv\mathsf{I}$. Suppose that (i) has been established for all J greater than or equal to some cardinality $\mathtt{d}$, and that J has cardinality $\mathtt{d\text{-}1}$. Let $\mathtt{p}$ be an element of $\mathsf{I}\triangle\mathsf{J}$. By our induction hypothesis we know that there exists $\mathtt{q}$ in $\overline{(\mathsf{J}\cup\{\mathtt{p}\})}$ such that $\theta(\mathtt{q}) = \psi(\mathtt{q})$. It follows from (2.10.7) and (2.10.8) that the meridian $\overleftrightarrow{\mathtt{p},\mathtt{q}}$ intersects $\boxed{\mathsf{J}}$ in a single point $\mathtt{j}$ which is in $\boxed{\mathsf{J}}$. We know that θ and ψ agree on both $\mathtt{p}$ and $\mathtt{q}$, whence follows that $\theta[\overleftrightarrow{\mathtt{p},\mathtt{q}}] = \psi[\overleftrightarrow{\mathtt{p},\mathtt{q}}]$. Since θ and ψ agree on J, it follows that $\theta[\boxed{\mathsf{J}}] = \psi[\boxed{\mathsf{J}}]$. It follows that $\theta(\mathtt{j})$ must equal $\psi(\mathtt{j})$. We have shown that (i) holds.

Suppose that $\{\mathtt{a},\mathtt{b}\}$ is any doubleton comprised by J. It follows from (i) that there exists some element $\mathtt{c}$ of $\overline{\{\mathtt{a},\mathtt{b}\}}$ on which θ and ψ agree. Let $\mathtt{t}$ be any element of $\overleftrightarrow{\mathtt{a},\mathtt{b}}$. The fact that θ is autographic implies that $\boxed{\begin{matrix}a & b & c & t\\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit\end{matrix}} \overset{\mathsf{P}}{\approx} \boxed{\begin{matrix}\theta(a) & \theta(b) & \theta(c) & \theta(t)\\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit\end{matrix}}$, which in turn implies that $\boxed{\begin{matrix}\theta(a) & \theta(b) & \theta(c)\\ a & b & c\end{matrix}}(\mathtt{t}) = \theta(\mathtt{t})$. But ψ agrees with θ on $\mathtt{a}$, $\mathtt{b}$ and $\mathtt{c}$ and $\boxed{\begin{matrix}\psi(a) & \psi(b) & \psi(c)\\ a & b & c\end{matrix}}(\mathtt{t}) = \psi(\mathtt{t})$ by the uniqueness postulate. Hence,

$$\theta(\mathtt{t}) = \psi(\mathtt{t}) \quad \text{for all } \mathtt{t}\in\overleftrightarrow{\mathtt{a},\mathtt{b}}\,. \tag{ii}$$

Suppose that $\mathtt{n}$ is such that, for all $\mathtt{m}$ less that or equal to $\mathtt{n}$, θ and ψ agree on $\boxed{\mathsf{J}}$ for each subset J of I such that the cardinality of J equals $\mathtt{m}$. We know from (ii) that $\mathtt{n}$ is at least $\mathtt{1}$. If $\mathtt{n\text{-}1}$ is the dimension of P, we know that θ and ψ on all of P, so we shall suppose otherwise: that J is a proper subset of I and that θ equals ψ on $\boxed{\mathsf{J}}$. From (i) we know that there exists an element $\mathtt{j}$ of $\boxed{\mathsf{J}}$ such that $\theta(\mathtt{j}) = \psi(\mathtt{j})$. Let $\mathtt{t}$ be any element of $\boxed{\mathsf{J}}$ distinct from $\mathtt{j}$. The meridian $\overleftrightarrow{\mathtt{j},\mathtt{t}}$ intersects at at least two separate points sets $\boxed{\mathsf{R}}$ and $\boxed{\mathsf{S}}$ where R and S are proper subsets of J. If $\mathtt{r}$ and $\mathtt{s}$ are those points, we know from our induction hypotheses that $\theta(\mathtt{r}) = \psi(\mathtt{s})$. We also know that $\theta(\mathtt{j}) = \psi(\mathtt{j})$. Since θ is autographic, the argument used to show (ii) shows that θ equals ψ everywhere on $\overleftrightarrow{\mathtt{j},\mathtt{t}}$: in particular, $\theta(\mathtt{t}) = \psi(\mathtt{t})$. That is, θ and ψ agree on all of J. Hence θ is just ψ, and so θ is a homography.

The von Staudt scalar functions of (2.11.13) are all isographies of meridian subsets of P onto the von Staudt meridian $\mathfrak{W}_{\mathsf{P}}$.

(2.13) Field Isomorphisms Let $[\infty,\mathfrak{o},\mathfrak{l}]$ be an ordered tripleton comprised by a meridian M, and let ϕ be a permutation of M which leaves ∞, $\mathfrak{o}$ and $\mathfrak{o}$ fixed. Let F be the field with ordered fundamental set $[\infty,\mathfrak{o},\mathfrak{l}]$. Then

$$\phi \text{ is a projective automorphism if, and only if, it is a field isomorphism.} \tag{1}$$

First suppose that ϕ is a field isomorphism. Let θ be any homography of M. Then there exist non-zero elements $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d}$ of F such that $\theta = \left(\begin{smallmatrix}\mathfrak{a} & \mathfrak{b}\\ \mathfrak{c} & \mathfrak{d}\end{smallmatrix}\right)$ where $\left(\begin{smallmatrix}\mathfrak{a} & \mathfrak{b}\\ \mathfrak{c} & \mathfrak{d}\end{smallmatrix}\right)$ is as in (2.2.2). Then, for each $\mathfrak{x}$ in F,

$$\phi\circ\theta\circ\phi^{-1}(\mathfrak{x}) \;=\; \phi\Big(\frac{\mathfrak{a}\cdot\phi^{-1}(\mathfrak{x})+\mathfrak{b}}{\mathfrak{c}\cdot\phi^{-1}(\mathfrak{x})+\mathfrak{d}}\Big) \;=\; \frac{\phi(\mathfrak{a})\cdot\mathfrak{x}+\phi(\mathfrak{b})}{\phi(\mathfrak{c})\cdot\mathfrak{x}+\phi(\mathfrak{d})} \;=\; \left(\begin{smallmatrix}\phi(\mathfrak{a}) & \phi(\mathfrak{b})\\ \phi(\mathfrak{c}) & \phi(\mathfrak{d})\end{smallmatrix}\right)(\mathfrak{x}).$$

Since $\left(\begin{smallmatrix}\phi(\mathfrak{a}) & \phi(\mathfrak{b})\\ \phi(\mathfrak{c}) & \phi(\mathfrak{d})\end{smallmatrix}\right)$ is a homography, it follows that ϕ is a projective automorphism.

Now we suppose, on the other hand, that ϕ is a projective automorphism. Let $\mathfrak{x}$ and $\mathfrak{y}$ be non-zero elements of F. We know from (2.7.5) that $\left(\begin{smallmatrix}\phi(\infty) & \phi(\mathfrak{x})\\ \phi(\infty) & \phi(\mathfrak{y})\end{smallmatrix}\right) = \phi\circ\left(\begin{smallmatrix}\infty & \mathfrak{x}\\ \infty & \mathfrak{y}\end{smallmatrix}\right)\circ\phi^{-1}$, whence follows

$$\phi(\mathfrak{x}+\mathfrak{y}) \overset{\text{by (2.8.10)}}{=\!=} \phi\Big(\left(\begin{smallmatrix}\infty & \mathfrak{x}\\ \infty & \mathfrak{y}\end{smallmatrix}\right)(\mathfrak{o})\Big) \;=\; \phi\circ\left(\begin{smallmatrix}\infty & \mathfrak{x}\\ \infty & \mathfrak{y}\end{smallmatrix}\right)\circ\phi^{-1}(\mathfrak{o}) \;=\; \left(\begin{smallmatrix}\phi(\infty) & \phi(\mathfrak{x})\\ \phi(\infty) & \phi(\mathfrak{y})\end{smallmatrix}\right)(\mathfrak{o}) \;=\; \left(\begin{smallmatrix}\infty & \phi(\mathfrak{x})\\ \infty & \phi(\mathfrak{y})\end{smallmatrix}\right)(\mathfrak{o}) \;=\; \phi(\mathfrak{x})+\phi(\mathfrak{y}).$$

We also know from (2.7.5) that $\left(\begin{smallmatrix}\phi(\infty) & \phi(\mathfrak{x})\\ \phi(\mathfrak{o}) & \phi(\mathfrak{y})\end{smallmatrix}\right) = \phi\circ\left(\begin{smallmatrix}\infty & \mathfrak{x}\\ \mathfrak{o} & \mathfrak{y}\end{smallmatrix}\right)\circ\phi^{-1}$, whence follows

$$\phi(\mathfrak{x}\cdot\mathfrak{y}) \overset{\text{by (2.8.10)}}{=\!=} \phi\Big(\left(\begin{smallmatrix}\infty & \mathfrak{x}\\ \mathfrak{o} & \mathfrak{y}\end{smallmatrix}\right)(\mathfrak{l})\Big) \;=\; \phi\circ\left(\begin{smallmatrix}\infty & \mathfrak{x}\\ \mathfrak{o} & \mathfrak{y}\end{smallmatrix}\right)\circ\phi^{-1}(\mathfrak{l}) \;=\; \left(\begin{smallmatrix}\phi(\infty) & \phi(\mathfrak{x})\\ \phi(\mathfrak{o}) & \phi(\mathfrak{y})\end{smallmatrix}\right)(\mathfrak{l}) \;=\; \left(\begin{smallmatrix}\infty & \phi(\mathfrak{x})\\ \mathfrak{o} & \phi(\mathfrak{y})\end{smallmatrix}\right)(\mathfrak{l}) \;=\; \phi(\mathfrak{x})\cdot\phi(\mathfrak{y}).$$

It follows that ϕ is a field isomophism.

(2.14) Rational Meridians There are some projective spaces on which all projective isomorphisms are isographic: we examine some of them now. Suppose that M is a meridian and that $\mathfrak{u}$ is an ordered tripleton in M. We shall write $\overset{\mathfrak{u}}{\infty}$, $\overset{\mathfrak{u}}{\mathfrak{o}}$ and $\overset{\mathfrak{u}}{\mathfrak{l}}$ for the elements of M such that $\mathfrak{u} = [\overset{\mathfrak{u}}{\infty},\overset{\mathfrak{u}}{\mathfrak{o}},\overset{\mathfrak{u}}{\mathfrak{l}}]$. Let $\overset{\mathfrak{u}}{+}$, $\overset{\mathfrak{u}}{-}$, $\overset{\mathfrak{u}}{\cdot}$ and $\overset{\mathfrak{u}}{/}$ be the field operations pertaining to the field associated with $\mathfrak{u}$ (*cf.* (2.8.10)). For any positive integer $\mathfrak{n}$, let $\overset{\mathfrak{u}}{\mathfrak{n}}$ denote the sum $\overbrace{\overset{\mathfrak{u}}{\mathfrak{l}}+\cdots+\overset{\mathfrak{u}}{\mathfrak{l}}}^{\mathfrak{n}\ \textit{times}}$. We shall say that M is a **$\mathfrak{u}$-rational meridian** if each element of M distinct from $\overset{\mathfrak{u}}{\infty}$ and from $\overset{\mathfrak{u}}{\mathfrak{o}}$ either is of the form $\overset{\mathfrak{u}}{\mathfrak{m}}\overset{\mathfrak{u}}{/}\overset{\mathfrak{u}}{\mathfrak{n}}$ or of the form $\overset{\mathfrak{u}}{-}\overset{\mathfrak{u}}{\mathfrak{m}}\overset{\mathfrak{u}}{/}\overset{\mathfrak{u}}{\mathfrak{n}}$ for positive integers $\mathfrak{m}$ and $\mathfrak{n}$.

Suppose that ψ is any projective isomormphism defined on a meridian M. Then, for any ordered tripleton $[\mathfrak{a},\mathfrak{b},\mathfrak{c}]$ in M and any ordered tripleton $[\mathfrak{x},\mathfrak{y},\mathfrak{z}]$ in the range of ψ, it follows from the uniqueness postulate that

$$\psi\circ\left(\begin{smallmatrix}\mathfrak{x} & \mathfrak{y} & \mathfrak{z}\\ \mathfrak{a} & \mathfrak{b} & \mathfrak{c}\end{smallmatrix}\right)\circ\psi^{-1} \;=\; \left(\begin{smallmatrix}\psi(\mathfrak{x}) & \psi(\mathfrak{y}) & \psi(\mathfrak{z})\\ \psi(\mathfrak{a}) & \psi(\mathfrak{b}) & \psi(\mathfrak{c})\end{smallmatrix}\right). \tag{1}$$

If M is a $\mathfrak{u}$-rational meridian and if $\mathfrak{v}$ is defined to be the ordered tripleton $[\psi(\overset{\mathfrak{u}}{\infty}),\psi(\overset{\mathfrak{u}}{\mathfrak{o}}),\psi(\overset{\mathfrak{u}}{\mathfrak{l}})]$, then

$$(\forall\,\mathfrak{m}\text{ and }\mathfrak{n}\text{ non-zero integers})\quad \psi(\overset{\mathfrak{u}}{\mathfrak{m}}\overset{\mathfrak{u}}{/}\overset{\mathfrak{u}}{\mathfrak{n}}) \;=\; \overset{\mathfrak{v}}{\mathfrak{m}}\overset{\mathfrak{v}}{/}\overset{\mathfrak{v}}{\mathfrak{n}}. \tag{2}$$

For the case $\mathfrak{m}=\mathfrak{n}=1$, we have $\overset{\mathfrak{u}}{\mathfrak{m}}\overset{\mathfrak{u}}{/}\overset{\mathfrak{u}}{\mathfrak{n}}=\overset{\mathfrak{u}}{\mathfrak{l}}$ and $\overset{\mathfrak{v}}{\mathfrak{m}}\overset{\mathfrak{v}}{/}\overset{\mathfrak{v}}{\mathfrak{n}}=\overset{\mathfrak{v}}{\mathfrak{l}}$, whence follows that (2) is trivial.

Suppose that we have established (2) for $\mathfrak{n}\equiv 1$ and all $\mathfrak{m}\le\mathfrak{k}$ for some integer $\mathfrak{k}$ greater than or equal to 1. Then, for $\mathfrak{m}\equiv\mathfrak{k}+1$ and $\mathfrak{n}\equiv 1$,

$$\psi(\overset{\mathfrak{u}}{\mathfrak{m}}\overset{\mathfrak{u}}{/}\overset{\mathfrak{u}}{\mathfrak{n}}) \;=\; \psi(\overbrace{\overset{\mathfrak{u}}{\mathfrak{l}}+\cdots+\overset{\mathfrak{u}}{\mathfrak{l}}}^{\mathfrak{k}+1\ \textit{times}}) \;=\; \psi(\overset{\mathfrak{u}}{\mathfrak{l}}+\overbrace{\overset{\mathfrak{u}}{\mathfrak{l}}+\cdots+\overset{\mathfrak{u}}{\mathfrak{l}}}^{\mathfrak{k}\ \textit{times}}) \overset{\text{by (2.8.10)}}{=\!=} \psi(\left(\begin{smallmatrix}\overset{\mathfrak{u}}{\infty} & \overset{\mathfrak{u}}{\mathfrak{k}}\\ \overset{\mathfrak{u}}{\infty} & \overset{\mathfrak{u}}{\mathfrak{l}}\end{smallmatrix}\right)(\overset{\mathfrak{u}}{\mathfrak{o}})) \;=$$

by our induction hypothesis

$$\psi(\left(\begin{smallmatrix}\psi^{-1}(\overset{\mathfrak{v}}{\infty}) & \psi^{-1}(\overset{\mathfrak{v}}{\mathfrak{k}})\\ \psi^{-1}(\overset{\mathfrak{v}}{\infty}) & \psi^{-1}(\overset{\mathfrak{v}}{\mathfrak{l}})\end{smallmatrix}\right)(\psi^{-1}(\overset{\mathfrak{v}}{\mathfrak{o}}))) \overset{\text{by (1)}}{=\!=} \left(\begin{smallmatrix}\overset{\mathfrak{v}}{\infty} & \overset{\mathfrak{v}}{\mathfrak{k}}\\ \overset{\mathfrak{v}}{\infty} & \overset{\mathfrak{v}}{\mathfrak{l}}\end{smallmatrix}\right)(\overset{\mathfrak{v}}{\mathfrak{o}}) \overset{\text{by (2.8.10)}}{=\!=} \overset{\mathfrak{v}}{\mathfrak{l}}+\overbrace{\overset{\mathfrak{v}}{\mathfrak{l}}+\cdots+\overset{\mathfrak{v}}{\mathfrak{l}}}^{\mathfrak{k}\ \textit{times}} = \overset{\mathfrak{v}}{\mathfrak{m}}\overset{\mathfrak{v}}{/}\overset{\mathfrak{v}}{\mathfrak{n}}.$$

Thus, by induction,

$$\text{we have established (2) for } \mathfrak{n} \equiv \mathtt{1}. \tag{i}$$

For general positive $\mathfrak{m}$ and $\mathfrak{n}$ we now have

$$\psi(\overset{\mathfrak{u}}{\mathfrak{m}}/\overset{\mathfrak{u}}{\mathfrak{n}}) \overset{\text{by (2.8.10)}}{=\!=\!=} \psi(\begin{bmatrix}\overset{\mathfrak{u}}{\infty} & \overset{\mathfrak{u}}{\mathfrak{m}} \\ \overset{\mathfrak{u}}{\mathfrak{o}} & \overset{\mathfrak{u}}{\mathfrak{l}}\end{bmatrix}(\overset{\mathfrak{u}}{\mathfrak{n}})) \overset{\text{by (i)}}{=\!=\!=} \psi(\begin{bmatrix}\psi^{-1}(\overset{\mathfrak{v}}{\infty}) & \psi^{-1}(\overset{\mathfrak{v}}{\mathfrak{m}}) \\ \psi^{-1}(\overset{\mathfrak{v}}{\mathfrak{o}}) & \psi^{-1}(\overset{\mathfrak{v}}{\mathfrak{l}})\end{bmatrix})(\psi^{-1}(\overset{\mathfrak{v}}{\mathfrak{n}})) \overset{\text{by (1)}}{=\!=\!=} \begin{bmatrix}\overset{\mathfrak{v}}{\infty} & \overset{\mathfrak{v}}{\mathfrak{m}} \\ \overset{\mathfrak{v}}{\mathfrak{o}} & \overset{\mathfrak{v}}{\mathfrak{l}}\end{bmatrix}(\overset{\mathfrak{v}}{\mathfrak{n}}) \overset{\text{by (2.8.10)}}{=\!=\!=} \overset{\mathfrak{v}}{\mathfrak{m}}/\overset{\mathfrak{v}}{\mathfrak{n}}.$$

It is evident that that (2) holds for negative integers as well.

If w is any other ordered tripleton in M and we set ψ equal to $\begin{bmatrix}\overset{\mathsf{w}}{\infty} & \overset{\mathsf{w}}{\mathfrak{o}} & \overset{\mathsf{w}}{\mathfrak{l}} \\ \overset{\mathsf{u}}{\infty} & \overset{\mathsf{u}}{\mathfrak{o}} & \overset{\mathsf{u}}{\mathfrak{l}}\end{bmatrix}$, it follows from (2) that

$$\mathsf{M} \text{ is } \mathsf{w}\text{-rational whenever it is } \mathsf{u}\text{-rational.} \tag{3}$$

Thus we may speak simply of a **rational meridian** if that meridian is u-rational for any ordered tripleton u.

It is also evident from (2) that

$$\text{the isomorphic image of a rational meridian is rational.} \tag{4}$$

Finally we note that,

$$\text{if } \mathsf{M} \text{ is rational, then each projective isomorphism } \theta \text{ on } \mathsf{M} \text{ is isographic.} \tag{5}$$

Let u be an ordered tripleton comprised by the rational meridian M. Let w be the ordered tripleton $[\theta(\overset{\mathsf{u}}{\infty}),\theta(\overset{\mathsf{u}}{\mathfrak{o}}),\theta(\overset{\mathsf{u}}{\mathfrak{l}})]$. Then $\psi \equiv \begin{bmatrix}\overset{\mathsf{u}}{\infty} & \overset{\mathsf{u}}{\mathfrak{o}} & \overset{\mathsf{u}}{\mathfrak{l}} \\ \overset{\mathsf{w}}{\infty} & \overset{\mathsf{w}}{\mathfrak{o}} & \overset{\mathsf{w}}{\mathfrak{l}}\end{bmatrix} \circ \theta$ is a projective isomorphism which leaves each element of u fixed. It follows from (2) that ψ is the identity mapping. Thus θ must equal $\begin{bmatrix}\overset{\mathsf{w}}{\infty} & \overset{\mathsf{w}}{\mathfrak{o}} & \overset{\mathsf{w}}{\mathfrak{l}} \\ \overset{\mathsf{u}}{\infty} & \overset{\mathsf{u}}{\mathfrak{o}} & \overset{\mathsf{u}}{\mathfrak{l}}\end{bmatrix}$, which is a homography.

(2.15) Harmonic Pairs Let M be a meridian. Let $\{\mathfrak{a},\mathfrak{b},\mathfrak{u},\mathfrak{v}\}$ be a quadrupleton in M. We shall say that $\{\{\mathfrak{a},\mathfrak{b}\},\{\mathfrak{u},\mathfrak{v}\}\}$ is a **harmonic pair** (**of pairs**) if

$$\begin{bmatrix}\mathfrak{u} & \mathfrak{b} & \mathfrak{v} \\ \mathfrak{a} & \mathfrak{u} & \mathfrak{b}\end{bmatrix}(\mathfrak{v}) = \mathfrak{a}\,. \tag{1}$$

If (1) holds

$$\begin{bmatrix}\mathfrak{u} & \mathfrak{b} & \mathfrak{v} \\ \mathfrak{a} & \mathfrak{u} & \mathfrak{b}\end{bmatrix} \text{ is then is of order } 4 \text{ as a permuitation group element.} \tag{2}$$

From (1) follows that $\begin{bmatrix}\mathfrak{u} & \mathfrak{b} & \mathfrak{v} \\ \mathfrak{a} & \mathfrak{u} & \mathfrak{b}\end{bmatrix} \circ \begin{bmatrix}\mathfrak{u} & \mathfrak{b} & \mathfrak{v} \\ \mathfrak{a} & \mathfrak{u} & \mathfrak{b}\end{bmatrix} \circ \begin{bmatrix}\mathfrak{u} & \mathfrak{b} & \mathfrak{v} \\ \mathfrak{a} & \mathfrak{u} & \mathfrak{b}\end{bmatrix} \circ \begin{bmatrix}\mathfrak{u} & \mathfrak{b} & \mathfrak{v} \\ \mathfrak{a} & \mathfrak{u} & \mathfrak{b}\end{bmatrix}$ agrees with the identity ι on each of the elements of the fundamental subset $\{\mathfrak{a},\mathfrak{u},\mathfrak{b}\}$ of M. It follows from the uniqueness postulate that the two homographies are identical.

Conversely, if ϕ is a homography on M such that ϕ is not an involution and such that $\phi\circ\phi\circ\phi\circ\phi$ is the identity homography ι on M, then

$$(\forall\, \mathfrak{x} \in \mathsf{M}\colon\ \phi(\mathfrak{x}) \neq \mathfrak{x}) \qquad \{\{\mathfrak{x},\phi\circ\phi(\mathfrak{x})\},\{\phi(\mathfrak{x}),\phi\circ\phi\circ\phi(\mathfrak{x})\}\} \text{ is a harmonic pair.} \tag{3}$$

If $\phi\circ\phi\circ\phi\circ\phi = \iota$ and we set $\mathfrak{a} \equiv \mathfrak{x}$, $\mathfrak{u} \equiv \phi(\mathfrak{x})$ $\mathfrak{b} \equiv \phi\circ\phi(\mathfrak{x})$ and $\mathfrak{v} \equiv \phi\circ\phi\circ\phi(\mathfrak{x})$ it would follow that (1) would hold, which means that (3) would hold.

A characterization of another sort is

$$\{\{\mathfrak{a},\mathfrak{b}\},\{\mathfrak{u},\mathfrak{v}\}\} \text{ is a harmonic pair} \iff \begin{bmatrix}\mathfrak{a} & \mathfrak{b} \\ \mathfrak{a} & \mathfrak{b}\end{bmatrix}(\mathfrak{u}) = \mathfrak{v}\,. \tag{4}$$

Suppose that $\{\{a,b\},\{u,v\}\}$ is a harmonic pair. Evidently $\left[\begin{smallmatrix} a & b \\ u & v \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} u & b & v \\ a & u & b \end{smallmatrix}\right]$ interchanges u and v and so is an involution. From (1) follows that $\left[\begin{smallmatrix} a & b \\ u & v \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} u & b & v \\ a & u & b \end{smallmatrix}\right]$ agrees with $\left[\begin{smallmatrix} a & b \\ a & b \end{smallmatrix}\right]$ at a and b. Since both are involutions which fix the same two points, it follows from (2.8.4) that they are identical. Thus $\left[\begin{smallmatrix} a & b \\ a & b \end{smallmatrix}\right](u) = v$.

Now suppose instead that $\left[\begin{smallmatrix} a & b \\ a & b \end{smallmatrix}\right](u) = v$. The involution $\left[\begin{smallmatrix} u & v \\ a & b \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & b \\ a & b \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} u & v \\ a & b \end{smallmatrix}\right]$ agrees with the involution $\left[\begin{smallmatrix} u & v \\ u & v \end{smallmatrix}\right]$ at both u and v and so they are identical. Thus $\left[\begin{smallmatrix} u & v \\ u & v \end{smallmatrix}\right]$ interchanges a and b and so $\left[\begin{smallmatrix} v & u \\ a & b \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} u & v \\ u & v \end{smallmatrix}\right]$ agrees with $\left[\begin{smallmatrix} u & b & v \\ a & u & b \end{smallmatrix}\right]$ at a, u and b. Hence $\left[\begin{smallmatrix} v & u \\ a & b \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} u & v \\ u & v \end{smallmatrix}\right] = \left[\begin{smallmatrix} u & b & v \\ a & u & b \end{smallmatrix}\right]$. Thus

$$\left[\begin{smallmatrix} u & b & v \\ a & u & b \end{smallmatrix}\right](v) \;=\; \left[\begin{smallmatrix} v & u \\ a & b \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} u & v \\ u & v \end{smallmatrix}\right](v) \;=\; \left[\begin{smallmatrix} v & u \\ a & b \end{smallmatrix}\right](v) \;=\; a\,.$$

If $\{\{a,b\},\{u,v\}\}$ is a harmonic pair, then

$$\left[\begin{smallmatrix} a & b & v \\ a & u & b \end{smallmatrix}\right] \text{ is a translation.} \tag{5}$$

By (2.8.6) we know that $\left[\begin{smallmatrix} a & b \\ a & b \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & b \\ a & u \end{smallmatrix}\right]$ is a translation. By (4) and direct computation we know that $\left[\begin{smallmatrix} a & b \\ a & b \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & b \\ a & u \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & b & v \\ a & u & b \end{smallmatrix}\right]$. This establishes (5).

Conversely, if τ is a translation of M, p is the fixed point of τ and if x is any element of M distinct from p, then

$$\{\{p,\tau(x)\},\{x,\tau\circ\tau(x)\}\} \text{ is a harmonic pair.} \tag{6}$$

It follows from (2.8.7) that

$$\tau = \left[\begin{smallmatrix} p & \tau(x) \\ p & \tau(x) \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} p & \tau(x) \\ p & x \end{smallmatrix}\right]. \tag{i}$$

We have

$$\left[\begin{smallmatrix} p & \tau(x) \\ p & \tau(x) \end{smallmatrix}\right](x) \;=\; \left[\begin{smallmatrix} p & \tau(x) \\ p & \tau(x) \end{smallmatrix}\right]\left(\left[\begin{smallmatrix} p & \tau(x) \\ p & x \end{smallmatrix}\right](\tau(x))\right) \;=\; \left[\begin{smallmatrix} p & \tau(x) \\ p & \tau(x) \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} p & \tau(x) \\ p & x \end{smallmatrix}\right](\tau(x)) \;\overset{\text{by (i)}}{=\!=\!=}\; \tau\circ\tau(x)\,. \tag{ii}$$

By (4), (ii) inplies (6).

It is sometimes the case that, given a tripleton $\{a,b,u\}$, one wants to express the element v such that $\{\{a,b\},\{u,v\}\}$ is a harmonic pair. That is, we want to find the **harmonic conjugate of u relative to** $\{a,b\}$. The solution is given by (4):

$$v \;=\; \left[\begin{smallmatrix} a & b \\ a & b \end{smallmatrix}\right](u)\,. \tag{7}$$

If $\{\{a,b\},\{u,v\}\}$ is a harmonic pair and ϕ is a projective isomorphism, then

$$\{\{\phi(a),\phi(b)\},\{\phi(u),\phi(v)\}\} \quad \text{is a harmonic pair.} \tag{8}$$

Since $\{\{a,b\},\{u,v\}\}$ is a harmonic pair, there exists a homography θ which sends a to u, u to b, b to v and v to a. Since ϕ is a projective isomorphism, we know that $\phi\circ\theta\circ\phi^{-1}$ is a homography. Evidently this homography sends $\phi(a)$ to $\phi(u)$, $\phi(u)$ to $\phi(b)$, $\phi(b)$ to $\phi(v)$ and $\phi(v)$ to $\phi(a)$. Consequently, (8) holds.

(2.16) Projections and Perspectivities Let P be a finite dimensional projective space of dimension greater than one, and let $\mathcal{P}$ be its homography group of permutations. If S is a maximal proper subspace, we say that S is a **co-point**.

Let p be a point in P and B a co-point not containing p. It follows from (2.10.8) that, for any point x distinct from p, the meridian $\overleftrightarrow{x,p}$ intersects B at one point. We shall denote this point by

$$\pi_{(p;B)}(x)\,. \tag{1}$$

The function $\pi_{(\mathsf{p};\mathsf{B})}$ thus induced will be called the **projection from** p **onto** B. If N is a meridian neither comprehended by B nor through p, the restriction of $\pi_{(\mathsf{p};\mathsf{B})}$ to N sends N onto a meridian within B: this function is called a **perspectivity**. Then

each perspectivity is a meridian isography. (2)

Let $\phi\big|\mathsf{N}\hookrightarrow\mathsf{M}$ be the restriction of a projection $\pi_{(\mathsf{p};\mathsf{B})}$ to a meridian N comprehended by P which neither contains p nor is comprehended by B. The sets $\{\mathsf{p}\}$, N and M are all comprehended by 2-dimensional subspace T of P. From (2.10.8) we know that N and M intersect at a single point q. Let $\{\mathsf{c},\mathsf{d}\}$ be a doubleton comprehended by N and not containing q. Let x be the intersection point of M with $\overleftrightarrow{\mathsf{p},\mathsf{c}}$ and y the intersection point of M with $\{\mathsf{p},\mathsf{d}\}$. The set $\{\mathsf{x},\mathsf{y},\mathsf{c},\mathsf{d}\}$ is evidently fundamental. By the fundamental postulate there exists an element ψ of $\mathcal{P}$ such that

$$\psi(\mathsf{c})=\mathsf{x}\,,\quad \psi(\mathsf{x})=\mathsf{c}\,,\quad \psi(\mathsf{d})=\mathsf{y}\,,\quad\text{and}\quad \psi(\mathsf{y})=\mathsf{d}\,. \tag{i}$$

Since ψ interchanges c with x and since $\overleftrightarrow{\mathsf{c},\mathsf{x}}$ is just $\overleftrightarrow{\mathsf{p},\mathsf{x}}$, ψ leaves the line $\overleftrightarrow{\mathsf{p},\mathsf{x}}$ invariant. Since ψ interchanges d with y and since $\overleftrightarrow{\mathsf{d},\mathsf{y}}$ is just $\overleftrightarrow{\mathsf{p},\mathsf{y}}$, ψ leaves the line $\overleftrightarrow{\mathsf{p},\mathsf{y}}$ invariant. It follows that the intersection of the lines $\overleftrightarrow{\mathsf{p},\mathsf{x}}$ and $\overleftrightarrow{\mathsf{p}}$ y is left invariant:

$$\psi(\mathsf{p})=\mathsf{p}\,. \tag{ii}$$

The meridian $\overleftrightarrow{\mathsf{c},\mathsf{d}}$ is sent by ψ onto the line $\overleftrightarrow{\mathsf{x},\mathsf{y}}$ and *vice-versa*: that is, ψ interchanges N and M. Thus the intersection point is fixed:

$$\psi(\mathsf{q})=\mathsf{q}\,. \tag{iii}$$

In view of (ii) and the fact that the points c and x are interchanged on $\overleftrightarrow{\mathsf{c},\mathsf{x}}$, it follows from (2.8.3) that property (2.1.4) that there is precisely one point o $\overleftrightarrow{\mathsf{p},\mathsf{x}}$ distinct from p such that

$$\psi(\mathsf{o})=\mathsf{o}\,. \tag{iv}$$

It follows from (iii) and (iv) that the line $\overleftrightarrow{\mathsf{q},\mathsf{o}}$ is sent onto itself by ψ and so the intersection r of $\overleftrightarrow{\mathsf{q},\mathsf{o}}$ with $\overleftrightarrow{\mathsf{p},\mathsf{y}}$ is sent to itself:

$$\psi(\mathsf{r})=\mathsf{r}\,. \tag{v}$$

From (iii), (iv) and (v) follows that ψ agrees with the identity function on the fundamental set $\{\mathsf{q},\mathsf{o},\mathsf{r}\}$, and so the uniqueness postulate implies that

$$(\forall\,\mathsf{z}\in\overleftrightarrow{\mathsf{q},\mathsf{o}})\quad \psi(\mathsf{z})=\mathsf{z}\,. \tag{vi}$$

Now let t be any element of N. The meridian $\overleftrightarrow{\mathsf{p},\mathsf{t}}$ intersects the meridian $\overleftrightarrow{\mathsf{q},\mathsf{o}}$ at some point u and intersects the meridian M at some point v. From (vi) follows that $\psi(\mathsf{u})=\mathsf{u}$. Thus $\overleftrightarrow{\mathsf{p},\mathsf{u}}$ is sent to itself by ψ and $\overleftrightarrow{\mathsf{p},\mathsf{u}}=\overleftrightarrow{\mathsf{p},\mathsf{v}}=\overleftrightarrow{\mathsf{p},\mathsf{t}}$. Since ψ interchanges the lines N and M, it follows that the intersection point v of $\overleftrightarrow{\mathsf{p},\mathsf{u}}$ with M is interchanged by ψ with the intersection point t of $\overleftrightarrow{\mathsf{p},\mathsf{u}}$ with N. We have

$$\psi(\mathsf{t})=\mathsf{v}=\pi_{(\mathsf{p};\mathsf{B})}(\mathsf{t})\,.$$

We have established that (2) holds.

(2.17) Libras A permutation group is a family of permutations of a set closed under compositions and containing the identity permutation. There are some applications however where one has such a family of permutations which does not contain the identity. We recall that a projective group on a meridian is generated by its involutions (*cf.* (2.9)). When constructing the fields of a meridian we made use of families of meridians which either interchanged two fixed points, or which kept one point fixed. These families were closed under composition (*cf.* (2.8.9)) but did not contain the identity function. In the sequel we shall have to deal with families similar to this, but whose common domain is not the same as their common range. This suggests we consider a construct more general than a group.

By a **transformation libra** we shall mean a family $\mathcal{F}$ of transformations with a common domain and a common range such that

$$(\forall\,\{\alpha,\beta,\gamma\}\subset\mathcal{F})\quad \gamma\circ\beta^{-1}\circ\alpha\in\mathcal{F}\,. \tag{1}$$

It is at times useful to introduce the following notation:

$$(\forall\,\{\alpha,\beta,\gamma\}\subset\mathcal{F})\quad [\![\gamma,\beta,\gamma]\!]\quad\equiv\quad \gamma\circ\beta^{-1}\circ\alpha\in\mathcal{F}\,. \tag{2}$$

Just as with a permutation group, which has an abstract form (abstract group), there is also an abstract

libra: a set L along with a trinary operation $\lfloor\ \rfloor\ \big|L\times L\times L\ni[x,y,z]\hookrightarrow\lfloor x,y,z\rfloor\in LF$ such that

$$(\forall\,\{x,y\}\subset L)\quad \lfloor x,y,y\rfloor=\lfloor y,y,x\rfloor=x\,, \tag{3}$$

and

$$(\forall\,\{v,y,x,w,z\}\subset L)\quad \lfloor\lfloor w,v,x\rfloor,y,z\rfloor=\lfloor w,v,\lfloor x,y,z\rfloor\rfloor\,. \tag{4}$$

It follows from (3) and (4) that the equality

$$\lfloor v,\lfloor w,x,y\rfloor,z\rfloor=\lfloor\lfloor v,y,x\rfloor,w,z\rfloor \tag{5}$$

holds as well.

We have

$$v\overset{\text{by (3)}}{=\!=}\lfloor v,w,w\rfloor\overset{\text{by (3)}}{=\!=}\lfloor v,w,\lfloor x,x,w\rfloor\rfloor\overset{\text{by (3)}}{=\!=}\lfloor v,w,\lfloor\lfloor x,y,y\rfloor,x,w\rfloor\rfloor\overset{\text{by (4)}}{=\!=}$$

$$\lfloor\lfloor v,w,\lfloor x,y,y\rfloor\rfloor,x,w\rfloor\overset{\text{by (4)}}{=\!=}\lfloor\lfloor\lfloor v,w,x\rfloor,y,y\rfloor,x,w\rfloor\overset{\text{by (4)}}{=\!=}\lfloor\lfloor v,w,x\rfloor,y,\lfloor y,x,w\rfloor\rfloor$$

whence follows that

$$\lfloor v,\lfloor y,x,w\rfloor,z\rfloor=\lfloor\lfloor\lfloor v,w,x\rfloor,y,\lfloor y,x,w\rfloor\rfloor,\lfloor y,x,w\rfloor,z\rfloor\overset{\text{by (4)}}{=\!=}$$
$$\lfloor\lfloor v,w,x\rfloor,y,\lfloor\lfloor y,x,w\rfloor,\lfloor y,x,w\rfloor,z\rfloor\rfloor\overset{\text{by (3)}}{=\!=}\lfloor\lfloor v,w,x\rfloor,y,z\rfloor\,.$$

A bijection from one libra onto another is said to be a **libra isomorphism** if it preserves the ternary libra operator.

A libra is said to be **abelian** provided that

$$((\forall\,\{x,y,z\}\subset L)\quad \lfloor x,y,z\rfloor=\lfloor z,y,x\rfloor\,. \tag{6}$$

A group G with binary operation $\cdot$ is a libra relative to the ternary operation

$$\lfloor x,y,z\rfloor\quad\equiv\quad x\cdot y^{-1}\cdot z\,. \tag{7}$$

Conversely, a libra L, relative to any element e of L, is a group relative to the operation

$$(\forall\,\{x,y\}\subset L)\quad x\cdot y\quad\equiv\quad\lfloor x,e,y\rfloor\,. \tag{8}$$

In this case we speak of the **libra group with identity** e.

Evidently a libra is abelian if, and only if, any one of the groups associated with it is abelian. In this case, all the groups associated with the libra are abelian.

(2.18) Involution Libras Let $\{a,b\}$ be a subset of a meridian M and let $\mathcal{M}$ be the homography group of M. We define

$$\mathcal{M}_{\{a,b\}}\quad\equiv\quad\{\phi\in\mathcal{M}:\phi\circ\phi=\iota\text{ and }\phi(a)=b\}\,. \tag{1}$$

Then

$$\mathcal{M}_{\{a,b\}}\text{ is an abelian libra.} \tag{2}$$

If a and b are distinct, it is evident that $\mathcal{M}_{\{a,b\}}$ is a libra. If they are identical, it follows from (2.8.9). That the meridian is abelian follows from the fact that we are dealing with involutions.

(2.19) Affine Space We recall that a maximal proper subspace of a finite dimensional projective space is called a **co-point**. Let O be such a co-point and let A be its complement $P\triangle O$. We say that A is the **affine space associated with** O.

If $\{a,c\}$ is a doubleton of A, then the meridian $\overleftrightarrow{a,c}$ intersects the co-point O in a single point

$$\infty_{\mathsf{O}}(\mathfrak{a},\mathfrak{c})\ . \tag{1}$$

We define the **midpoint between** $\mathfrak{a}$ and $\mathfrak{c}$ to be

$$\mathfrak{m} \ \equiv\ \left[\begin{smallmatrix}\mathfrak{a}&\mathfrak{c}\\ \mathfrak{a}&\mathfrak{c}\end{smallmatrix}\right](\infty_{\mathsf{O}}(\mathfrak{a},\mathfrak{c}))\ . \tag{2}$$

Thus the midpoint is the unique element $\mathfrak{m}$ of A such that $\{\mathfrak{a},\mathfrak{c}\}$ and $\{\mathfrak{m},\infty_{\mathsf{O}}(\mathfrak{a},\mathfrak{c})\}$ are harmonic pairs. The midpoint between a singleton $\mathfrak{a}$ and itself is defined to be $\mathfrak{a}$.

If $\mathfrak{b}$ is another point of A distinct from the midpoint $\mathfrak{m}$ between $\mathfrak{a}$ and $\mathfrak{c}$, then the **affine libra product**

$$\lfloor\mathfrak{a},\mathfrak{b},\mathfrak{c}\rfloor_{\mathsf{A}} \tag{3}$$

is defined to be

$$\left[\begin{smallmatrix}\mathfrak{b}&\mathfrak{m}\\ \mathfrak{b}&\mathfrak{m}\end{smallmatrix}\right](\infty_{\mathsf{O}}(\mathfrak{b},\mathfrak{m}))\ . \tag{4}$$

If $\mathfrak{b}$ is identical with $\mathfrak{m}$, the affine libra product is defined to be just $\mathfrak{b}$. The product $\lfloor\mathfrak{a},\mathfrak{b},\mathfrak{c}\rfloor_{\mathsf{A}}$ may be regarded as the element of A such that the points $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and $\lfloor\mathfrak{a},\mathfrak{b},\mathfrak{c}\rfloor_{\mathsf{A}}$ form the vertices of a "parallelogram".

Suppose that S is any co-point of P which does not contain $\mathfrak{p}$. Then he projection $\pi_{(\mathfrak{p};\mathsf{S})}$ from $\mathfrak{p}$ onto S preserves the ternary operation defined by (3):

$$(\forall\,\{\mathfrak{a},\mathfrak{b},\mathfrak{c}\}\subset\mathsf{A})\quad \pi_{(\mathfrak{p};\mathsf{S})}(\lfloor\mathfrak{a},\mathfrak{b},\mathfrak{c}\rfloor) = \lfloor\pi_{(\mathfrak{p};\mathsf{S})}(\mathfrak{a}),\pi_{(\mathfrak{p};\mathsf{S})}(\mathfrak{b}),\pi_{(\mathfrak{p};\mathsf{S})}(\mathfrak{c})\rfloor\ . \tag{5}$$

Let $\mathfrak{m}$ be as in (2). We know from (2.16.2) that the restriction μ of $\pi_{(\mathfrak{p};\mathsf{S})}$ to $\overleftrightarrow{\mathfrak{a},\mathfrak{c}}$ is a meridian isography onto the meridian $\overleftrightarrow{\pi_{(\mathfrak{p};\mathsf{S})}(\mathfrak{a}),\pi_{(\mathfrak{p};\mathsf{S})}(\mathfrak{c})}$. Since by (2.15.8) $\pi_{(\mathfrak{p};\mathsf{S})}$ preserves harmonic pairs, if $\mathfrak{m}$ is the midpoint of $\mathfrak{a}$ and $\mathfrak{c}$ in A, it follows that

$$\pi_{(\mathfrak{p};\mathsf{S})}(\mathfrak{m}) \text{ is the midpoint of } \pi_{(\mathfrak{p};\mathsf{S})}(\mathfrak{a}) \text{ and } \pi_{(\mathfrak{p};\mathsf{S})}(\mathfrak{c})\ . \tag{i}$$

We know from (2.16.2) that the restriction ν of $\pi_{(\mathfrak{p};\mathsf{S})}$ to $\overleftrightarrow{\mathfrak{b},\mathfrak{m}}$ is a meridian isography onto the meridian $\overleftrightarrow{\pi_{(\mathfrak{p};\mathsf{S})}(\mathfrak{b}),\pi_{(\mathfrak{p};\mathsf{S})}(\mathfrak{m})}$. Since by (2.15.8) $\pi_{(\mathfrak{p};\mathsf{S})}$ preserves harmonic pairs, it follows from (i) that

$$\pi_{(\mathfrak{p};\mathsf{S})}(\pi_{(\mathfrak{p};\mathsf{S})}(\mathfrak{m})) \text{ is the midpoint of } \pi_{(\mathfrak{p};\mathsf{S})}(\mathfrak{b}) \text{ and } \pi_{(\mathfrak{p};\mathsf{S})}(\infty_{\mathsf{O}}(\mathfrak{b},\mathfrak{m}))\ . \tag{ii}$$

From (i) and (ii) follows that (5) holds.

We may deduce in part from this that, relative to (3),

$$\mathsf{A} \text{ is an abelian libra.} \tag{6}$$

It is evident from the definition that the operator (3) is abelian. We first consider the case where P is one dimensional. In this case O is a point which we shall denote ∞. Let $\{\mathfrak{a},\mathfrak{o},\mathfrak{c}\}$ be any tripleton distinct from ∞. We know from (2.8.10) that A is a group with identity $\mathfrak{o}$ relative to the binary operation defined by $\mathfrak{a}+\mathfrak{c} \equiv \left[\begin{smallmatrix}\infty&\mathfrak{a}\\ \infty&\mathfrak{c}\end{smallmatrix}\right](\mathfrak{o})$. Let $\mathfrak{m} \equiv \left[\begin{smallmatrix}\mathfrak{a}&\mathfrak{c}\\ \mathfrak{a}&\mathfrak{c}\end{smallmatrix}\right](\infty)$. From (2.15.4) we know that $\{\{\mathfrak{a},\mathfrak{c}\},\{\mathfrak{m},\infty\}\}$ is a harmonic pair. It follows that $\left[\begin{smallmatrix}\infty&\mathfrak{m}\\ \infty&\mathfrak{m}\end{smallmatrix}\right]$ interchanges $\mathfrak{a}$ and $\mathfrak{c}$. Thus $\left[\begin{smallmatrix}\infty&\mathfrak{a}\\ \infty&\mathfrak{c}\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\infty&\mathfrak{m}\\ \infty&\mathfrak{m}\end{smallmatrix}\right]$ agree on the fundamental set $\{\mathfrak{a},\mathfrak{c},\infty\}$. It follows from the uniqueness postulate that they must be equal. Consequently

$$\lfloor\mathfrak{a},\mathfrak{o},\mathfrak{c}\rfloor\ =\ \mathfrak{a}+\mathfrak{c}\ . \tag{i}$$

By (2.8.10) we know that A is a group with identity $\mathfrak{o}$, whence follows that A is the corresponding group libra. Let $\mathfrak{n}$ be a positive integer and suppose that we have shown that (2.17.4) holds for A for all dimensions of P less than $\mathfrak{n}$. Suppose that P is of dimension $\mathfrak{n}$ and let $\{v,w,x,y,z\}$ be a subset of A. Assume that

$$\lfloor\lfloor v,w,x\rfloor,y,z\rfloor \neq \lfloor v,w,\lfloor x,y,z\rfloor\rfloor\ . \tag{ii}$$

Then we could find a proper subspace S of P containing both $\lfloor\lfloor v,w,x\rfloor,y,z\rfloor$ and $\lfloor v,w,\lfloor x,y,z\rfloor\rfloor$. It follows from (5) that

$\lfloor\lfloor \pi_{(p;S)}(v),\pi_{(p;S)}(w),\pi_{(p;S)}(x)\rfloor,\pi_{(p;S)}(y),\pi_{(p;S)}(z)\rfloor \overset{\text{by (5)}}{=\!=} \pi_{(p;S)}(\lfloor\lfloor v,w,x\rfloor,y,z\rfloor) = \lfloor\lfloor v,w,x\rfloor,y,z\rfloor \neq$ by (ii)

$\lfloor v,w,\lfloor x,y,z\rfloor\rfloor = \pi_{(p;S)}(\lfloor v,w,\lfloor x,y,z\rfloor\rfloor) \overset{\text{by (5)}}{=\!=} \lfloor \pi_{(p;S)}(v),\pi_{(p;S)}(w),\lfloor \pi_{(p;S)}(x),\pi_{(p;S)}(y),\pi_{(p;S)}(z)\rfloor\rfloor$

which were absurd by our induction hypothesis.

Let θ be an element of $\mathcal{P}$ which leaves O invariant. Then, for all $\{a,b,c\}$,

$$\theta(\lfloor a,b,c\rfloor) \;=\; \lfloor\theta(a),\theta(b),\theta(c)\rfloor\,. \tag{7}$$

This follows from (2.15.8).

(2.20) Lemma For our treatment of linear space below, we shall require the following identities for elements a, b, c, d and e of a meridian:

$$\left[\begin{smallmatrix} a & b & d \\ a & b & c \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & b & c \\ a & b & e \end{smallmatrix}\right] \;=\; \left[\begin{smallmatrix} a & b & c \\ a & b & e \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & b & d \\ a & b & c \end{smallmatrix}\right] \tag{1}$$

and

$$\left[\begin{smallmatrix} a & b & d \\ a & b & c \end{smallmatrix}\right](e) \;=\; \left[\begin{smallmatrix} a & b & e \\ a & b & c \end{smallmatrix}\right](d) \tag{2}$$

Direct computation shows that

$$\left[\begin{smallmatrix} a & b & d \\ a & b & c \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & d \\ b & c \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & c \\ b & c \end{smallmatrix}\right] \text{ and } \left[\begin{smallmatrix} a & b & c \\ a & b & e \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & c \\ b & e \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & e \\ b & e \end{smallmatrix}\right]. \tag{i}$$

By (2.18.2) we know that $\mathcal{M}_{\{a,b\}}$ is an abelian libra. Thus

$$\left[\begin{smallmatrix} a & b & d \\ a & b & c \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & b & c \\ a & b & e \end{smallmatrix}\right] \overset{\text{by (i)}}{=\!=} \left[\begin{smallmatrix} a & d \\ b & c \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & c \\ b & c \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & c \\ b & e \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & e \\ b & e \end{smallmatrix}\right] = \mathbb{L}\left[\begin{smallmatrix} a & d \\ b & c \end{smallmatrix}\right],\left[\begin{smallmatrix} a & c \\ b & c \end{smallmatrix}\right],\left[\begin{smallmatrix} a & c \\ b & e \end{smallmatrix}\right]\mathbb{J} \circ \left[\begin{smallmatrix} a & e \\ b & e \end{smallmatrix}\right] = \mathbb{L}\left[\begin{smallmatrix} a & c \\ b & e \end{smallmatrix}\right],\left[\begin{smallmatrix} a & c \\ b & c \end{smallmatrix}\right],\left[\begin{smallmatrix} a & d \\ b & c \end{smallmatrix}\right]\mathbb{J} \circ \left[\begin{smallmatrix} a & e \\ b & e \end{smallmatrix}\right] =$$

$$\left[\begin{smallmatrix} a & c \\ b & e \end{smallmatrix}\right] \circ \mathbb{L}\left[\begin{smallmatrix} a & c \\ b & c \end{smallmatrix}\right],\left[\begin{smallmatrix} a & d \\ b & c \end{smallmatrix}\right],\left[\begin{smallmatrix} a & e \\ b & e \end{smallmatrix}\right]\mathbb{J} = \left[\begin{smallmatrix} a & c \\ b & e \end{smallmatrix}\right] \circ \mathbb{L}\left[\begin{smallmatrix} a & e \\ b & e \end{smallmatrix}\right],\left[\begin{smallmatrix} a & d \\ b & c \end{smallmatrix}\right],\left[\begin{smallmatrix} a & c \\ b & c \end{smallmatrix}\right]\mathbb{J} = \left[\begin{smallmatrix} a & c \\ b & e \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & e \\ b & e \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & d \\ b & c \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & c \\ b & c \end{smallmatrix}\right] \overset{\text{by (i)}}{=\!=} \left[\begin{smallmatrix} a & b & c \\ a & b & e \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & b & d \\ a & b & c \end{smallmatrix}\right]$$

which yields (1).

Noting that $\left[\begin{smallmatrix} a & b & e \\ a & b & c \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} a & b & c \\ a & b & e \end{smallmatrix}\right]$, we have

$$\left[\begin{smallmatrix} a & b & d \\ a & b & c \end{smallmatrix}\right](e) = \left[\begin{smallmatrix} a & b & e \\ a & b & c \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & b & c \\ a & b & e \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & b & d \\ a & b & c \end{smallmatrix}\right](e) \overset{\text{by (1)}}{=\!=} \left[\begin{smallmatrix} a & b & e \\ a & b & c \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & b & d \\ a & b & c \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & b & c \\ a & b & e \end{smallmatrix}\right](e) = \left[\begin{smallmatrix} a & b & e \\ a & b & c \end{smallmatrix}\right](d)$$

which yields (2).

(2.21) Projective Linear Space Let P be a finite dimensional projective space, O a co-point of O, A the set-theoretic complement of O and o an element of A. We know from (2.19.6) that A is an affine space and a libra. The group associated with o is called a **linear space**. The group operation of (2.17.8) we shall denote by $+$: thus

$$(\forall\,\{a,b\}\subset A) \quad a+b \;\equiv\; \lfloor a,o,b\rfloor\,. \tag{1}$$

We recall the Von Staudt meridian $\mathfrak{W}_P$, the special elements $\boldsymbol{\infty}$, $\mathbf{0}$ and $\mathbf{1}$ of $\mathfrak{W}_P$, and the projective field $\mathfrak{F}_P$ of $\mathfrak{W}_P$ with ordered fundamental set $[\boldsymbol{\infty},\mathbf{0},\mathbf{1}]$. For any scalar $\mathbf{r}$ in $\mathfrak{F}_P$ and any element x of P, we define scalar multiplication as in (2.11.20):

$$\mathbf{r}\bullet x \quad\equiv\quad \begin{cases} o\,, & \text{if } \mathbf{r}=\mathbf{0} \text{ or } x=o\,; \\ \left[\begin{smallmatrix} \infty & o & x \\ \boldsymbol{\infty} & \mathbf{0} & \mathbf{1} \end{smallmatrix}\right](\mathbf{r})\,, & \text{for } x\in A\,,\ \mathbf{r}\neq\mathbf{0} \text{ and } x\neq o; \\ x\,, & \text{for } x\in O \text{ and } \mathbf{r}\neq\mathbf{0}\,. \end{cases} \tag{2}$$

For any non-zero scalar $\mathbf{r}$, the function

$$\mathbf{r}\bullet\big|P \ni x \hookrightarrow \mathbf{r}\bullet x \in P \text{ is a homography.} \tag{3}$$

Let M be any meridian comprehended by P containing o. By (2.10.8) the meridian M intersects O at a single point which we shall denote by ∞. Let $\mathfrak{l}$ be any element of $(\mathsf{M}\triangle\{\mathsf{o},\infty\})$. There is a unique element $\mathfrak{r}$ of M such that $\begin{smallmatrix}\infty&\mathsf{o}&\mathfrak{l}&\mathfrak{r}\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit\end{smallmatrix}\in\mathbf{r}$. For each $\mathfrak{x}$ in M

$$\mathbf{r}\bullet\mathfrak{x}\overset{\text{by (2)}}{=\!=}\begin{smallmatrix}\infty&\mathsf{o}&\mathfrak{x}\\ \infty&\mathsf{o}&\mathfrak{l}\end{smallmatrix}(\mathbf{r})=\begin{smallmatrix}\infty&\mathsf{o}&\mathfrak{x}\\ \infty&\mathsf{o}&\mathfrak{l}\end{smallmatrix}\circ\begin{smallmatrix}\infty&\mathsf{o}&\mathfrak{l}\\ \infty&\mathsf{o}&1\end{smallmatrix}(\mathbf{r})\overset{\text{by (2)}}{=\!=}\tag{i}$$

$$\begin{smallmatrix}\infty&\mathsf{o}&\mathfrak{x}\\ \infty&\mathsf{o}&\mathfrak{l}\end{smallmatrix}(\mathbf{r}\bullet\mathfrak{l})\overset{\text{by (2)}}{=\!=}\begin{smallmatrix}\infty&\mathsf{o}&\mathfrak{x}\\ \infty&\mathsf{o}&\mathfrak{l}\end{smallmatrix}(\mathfrak{r})\overset{\text{by (2)}}{=\!=}\begin{smallmatrix}\infty&\mathsf{o}&\mathfrak{r}\\ \infty&\mathsf{o}&\mathfrak{l}\end{smallmatrix}(\mathfrak{x})$$

It follows that (3) holds when P is one dimensional.

We now presume that P has dimension greater than 1. Let B be a maximal independent subset of O. It follows from (2.10.8) that $(\mathsf{B}\cup\{\mathsf{o}\})$ is a maximal independent subset of P. We know from (2.10.7) that $\overline{\mathsf{B}\cup\{\mathsf{o}\}}$ is non-void. Let $\mathfrak{l}$ be an element of $\overline{\mathsf{B}\cup\{\mathsf{o}\}}$. Thus $(\mathsf{B}\cup\{\mathsf{o},\mathfrak{l}\})$ is a fundamental set. Evidently $(\mathsf{B}\cup\{\mathsf{o},\mathbf{r}\bullet\mathfrak{l}\})$ is also fundamental. It follows from the fundamental postulate that there exists ϕ in $\mathcal{P}$ such that

$$\phi(\mathfrak{l})=\mathbf{r}\bullet\mathfrak{l}\quad\text{and}\quad(\forall\,\mathfrak{x}\in(\mathsf{B}\cup\{\mathsf{o}\}))\quad\phi(\mathfrak{x})=\mathfrak{x}\,.\tag{ii}$$

We shall show that

$$(\forall\,\mathfrak{x}\in\mathsf{A})\quad\phi(\mathfrak{x})\;=\;\mathbf{r}\bullet\mathfrak{x}\,.\tag{iii}$$

Let now M be $\overleftrightarrow{\mathsf{o},\mathfrak{l}}$ and let $\mathfrak{r}$ in M be such that $\begin{smallmatrix}\infty&\mathsf{o}&\mathfrak{l}&\mathfrak{r}\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit\end{smallmatrix}\in\mathbf{r}$. Since ϕ agrees with $\begin{smallmatrix}\infty&\mathsf{o}&\mathfrak{r}\\ \infty&\mathsf{o}&\mathfrak{l}\end{smallmatrix}$ on o $\mathfrak{l}$ and ∞, it follows from the uniqueness postulate that they agree on M. Consequently, in view of (i), we know that (iii) holds for all $\mathfrak{x}$ in M. Suppose that $\mathfrak{x}$ is not in M. Choose a co-point S containing o and $\mathfrak{x}$ but not $\mathfrak{l}$. Let p be the point where $\overleftrightarrow{\mathfrak{x},\mathfrak{l}}$ intersects S. It follows from (2.16.2) that the restriction μ to M of $\pi_{(\mathsf{p};\mathsf{S})}$ is a meridian isomorphism onto $\overleftrightarrow{\mathsf{o},\mathfrak{x}}$. In fact, if we denote by w the intersection point of $\overleftrightarrow{\mathsf{o},\mathfrak{x}}$ with S, we have

$$\mu\;=\;\begin{smallmatrix}w&\mathsf{o}&\mathfrak{x}\\ \infty&\mathsf{o}&\mathfrak{l}\end{smallmatrix}\,.\tag{iv}$$

It follows from (ii) that ϕ fixes both w and o, whence follows that ϕ leaves $\overleftrightarrow{\mathsf{o},w}$ invariant. Consequently the intersection point $\mathfrak{x}$ of the meridians $\overleftrightarrow{\mathfrak{v},\mathfrak{l}}$ and $\overleftrightarrow{\mathsf{o},w}$ is mapped by ϕ to the intersection point of the meridians $\overleftrightarrow{\mathfrak{v},\mathfrak{r}}$ and $\overleftrightarrow{\mathsf{o}}\,\mathsf{w}$. This intersection point is just $\pi_{(\mathsf{p};\mathsf{S})}(\mathfrak{r})$: that is,

$$\phi(\mathfrak{x})\;=\;\pi_{(\mathsf{p};\mathsf{S})}(\mathfrak{r})\overset{\text{by (iv)}}{=\!=}\begin{smallmatrix}w&\mathsf{o}&\mathfrak{x}\\ \infty&\mathsf{o}&\mathfrak{l}\end{smallmatrix}(\mathfrak{r})\,.\tag{v}$$

Since $\begin{smallmatrix}\infty&\mathsf{o}&\mathfrak{l}&\mathfrak{r}\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit\end{smallmatrix}$ is in $\mathbf{r}$, it follows that $\begin{smallmatrix}w&\mathsf{o}&\mathfrak{x}\\ \infty&\mathsf{o}&\mathfrak{l}\end{smallmatrix}\circ\begin{smallmatrix}\infty&\mathsf{o}&\mathfrak{l}&\mathfrak{r}\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit\end{smallmatrix}$ is in $\mathbf{r}$ as well. By (v) this last is just $\begin{smallmatrix}w&\mathsf{o}&\mathfrak{x}&\phi(\mathfrak{r})\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit\end{smallmatrix}$. It follows from (2.11.21) that (iii) holds for $\mathfrak{x}$.

Let $\mathbf{r}$ and $\mathbf{s}$ be elements of $\mathfrak{F}_{\mathsf{P}}$ and let $\mathfrak{x}$ be an element of A. Then

$$\mathbf{r}\bullet(\mathbf{s}\bullet\mathfrak{x})\;=\;(\mathbf{r}\cdot\mathbf{s})\bullet\mathfrak{x}\,.\tag{4}$$

We may presume that $\mathfrak{x}$ is not o and let w be the intersection of $\overleftrightarrow{\mathsf{o},w}$ with O. Let $\mathfrak{r}$ and $\mathfrak{s}$ be the elements of $\overleftrightarrow{\mathsf{o},w}$ such that

$$\begin{smallmatrix}w&\mathsf{o}&\mathfrak{l}&\mathfrak{r}\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit\end{smallmatrix}\in\mathbf{r}\quad\text{and}\quad\begin{smallmatrix}w&\mathsf{o}&\mathfrak{l}&\mathfrak{s}\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit\end{smallmatrix}\in\mathbf{s}\,.\tag{i}$$

We have

$$\mathbf{r}\bullet(\mathbf{s}\bullet\mathfrak{x})\overset{\text{by (2)}}{=\!=}\begin{smallmatrix}w&\mathsf{o}&\mathfrak{x}\\ \infty&\mathsf{o}&1\end{smallmatrix}(\mathbf{r}\cdot\mathbf{s})\overset{\text{by (2.11.19)}}{=\!=\!=}\begin{smallmatrix}w&\mathsf{o}&\mathfrak{x}\\ \infty&\mathsf{o}&1\end{smallmatrix}(\mathbf{r})\cdot\begin{smallmatrix}w&\mathsf{o}&\mathfrak{x}\\ \infty&\mathsf{o}&1\end{smallmatrix}(\mathbf{s})\overset{\text{by (2.11.20)}}{=\!=\!=}\mathfrak{r}\cdot\mathfrak{x}\cdot\mathfrak{s}\cdot\mathfrak{x}$$

If $\mathbf{1}$ is the multiplicative identity for $\mathfrak{W}_{\mathsf{P}}$ and $\mathfrak{x}$ is in $(\mathsf{A}\triangle\{\mathsf{o}\})$, then

$$\mathbf{1}\bullet\mathfrak{x}\;=\mathfrak{x}\,.\tag{5}$$

Let w denote the intersection point of $\overleftrightarrow{\mathsf{o},\mathfrak{x}}$ and O. It follows from (2.11.12) that $\begin{smallmatrix}w&\mathsf{o}&\mathfrak{x}&\mathfrak{x}\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit\end{smallmatrix}$ is in $\mathbf{1}$, so

$$\mathbf{1}\bullet\mathfrak{x}\overset{\text{by (2)}}{=\!=}\begin{smallmatrix}w&\mathsf{o}&\mathfrak{x}\\ \infty&\mathsf{o}&1\end{smallmatrix}(\mathbf{r}\cdot\mathbf{s})$$

Furthermore

$$(\mathbf{r}+\mathbf{s})\bullet\mathfrak{x}\;=\;(\mathbf{r}\bullet\mathfrak{x})+(\mathbf{s}\bullet\mathfrak{x})\,.\tag{6}$$

We have

$$(\mathbf{r}+\mathbf{s})\bullet\mathsf{x} \overset{\text{by (2)}}{=\!=\!=} \boxed{\begin{smallmatrix}w & o & x\\ 0 & 0 & 1\end{smallmatrix}}\,[\mathbf{r}+\mathbf{s}] \overset{\text{by (2.11.19)}}{=\!=\!=\!=} \boxed{\begin{smallmatrix}w & o & x\\ 0 & 0 & 1\end{smallmatrix}}\,[\mathbf{r}]+\boxed{\begin{smallmatrix}w & o & x\\ 0 & 0 & 1\end{smallmatrix}}\,[\mathbf{r}] \overset{\text{by (2)}}{=\!=\!=} (\mathbf{r}\bullet\mathsf{x})+(\mathbf{s}\bullet\mathsf{x})\,.$$

Finally, for y in A also distinct from o,

$$\mathbf{r}\bullet(\mathsf{x}+\mathsf{y}) \;=\; (\mathbf{r}\bullet\mathsf{x})+(\mathbf{r}\bullet\mathsf{y})\,. \tag{7}$$

We have

$$\mathbf{r}\bullet(\mathsf{x}+\mathsf{y}) \overset{\text{by (1)}}{=\!=\!=} \mathbf{r}\bullet\lfloor\mathsf{x},\mathsf{o},\mathsf{y}\rfloor \overset{\text{by (3) and (2.19.7)}}{=\!=\!=\!=\!=\!=\!=\!=\!=} \lfloor\mathbf{r}\bullet\mathsf{x},\mathbf{r}\bullet\mathsf{o},\mathbf{r}\bullet\mathsf{y}\rfloor \overset{\text{by (1)}}{=\!=\!=} (\mathbf{r}\bullet\mathsf{x})+(\mathbf{r}\bullet\mathsf{y})\,.$$

We have shown that the definition of linear space as given here is identical with the classical definition.

Section 3. Real Projective Space

(3.1) Introduction In analysis the projective spaces which appear most often are the **real projective spaces**: those of which the meridians are the real line with one point superadded or, equivalently, those isomorphic to the circle meridian.

In this section we investigate why this is so. A circle comprises its arcs and the real line its intervals. These arcs or intervals are inherent in the projective structure, and it is these which give it order and an intrinsic topology.

(3.2) Cycles E. V. Huntington in 1916 introduced the idea of a cyclic order. The idea is that of a set S arranged in order along a circle. To fix ideas, we shall imagine the direction as being counter-clockwise. In this context an ordered triple $[a,b,c]$ is associated with three points which may be thought of as arranged such that to get from a to c one must pass counter-clockwise through b. To formalize this idea we consider the set S_3 of all ordered tripletons of S and identify a subset $\curvearrowleft$ of S_3 such that, for each tripleton $\{a,b,c\}$ subsumed by S,

(Axiom I) exactly one of the ordered triples $[a,b,c]$ and $[c,b,a]$ is in $\curvearrowleft$ (1)

and

(Axiom II) $[b,c,a]$ is in $\curvearrowleft$ whenever $[a,b,c]$ is in $\curvearrowleft$. (2)

If an ordered tripleton $[a,b,c]$ is in $\curvearrowleft$, we shall write

$$c \curvearrowleft b \curvearrowleft a \tag{3}$$

and if it is not we shall write

$$a \curvearrowright b \curvearrowright c. \tag{4}$$

Given elements a and c of S, one may consider the "arc" or "interval" of elements through which one must pass to get to c from a (moving counter-clockwise). We define

$$(\overset{\curvearrowleft}{b,a}) \equiv \{x \in S : [a,x,b] \in \curvearrowleft.\} \tag{5}$$

If an element b of S_3 is in an arc $(\overset{\curvearrowleft}{c,a})$, we should expect that each element x of the arc $(\overset{\curvearrowleft}{b,a})$ would also be in the arc $(\overset{\curvearrowleft}{c,a})$. This suggests the third and final axiom for a **cyclic order $\curvearrowleft$ of S**:

(Axiom III) if $c \curvearrowleft b \curvearrowleft a$ and $d \curvearrowleft c \curvearrowleft a$ for $\{a,b,c,d\} \subset S$, then $d \curvearrowleft b \curvearrowleft a$. (6)

If all three axioms I, II and III hold we say that $\curvearrowleft$ is a **cyclic order** and that S is a **cycle**.

The hypotheses for (6) in any given cycle actually determine all possible relations among tripletons comprehended by $\{a,b,c,d\}$:

$$\text{if} \quad c \curvearrowleft b \curvearrowleft a \quad \text{and} \quad d \curvearrowleft c \curvearrowleft a, \tag{7}$$

then

$$a \curvearrowleft c \curvearrowleft b, \quad b \curvearrowleft a \curvearrowleft c, \quad a \curvearrowleft d \curvearrowleft c, \quad c \curvearrowleft a \curvearrowleft d, \quad d \curvearrowleft b \curvearrowleft a, \quad a \curvearrowleft d \curvearrowleft b, \quad b \curvearrowleft a \curvearrowleft d, \tag{8}$$

and

$$d \curvearrowleft c \curvearrowleft b, \quad b \curvearrowleft d \curvearrowleft c \quad \text{and} \quad c \curvearrowleft b \curvearrowleft d. \tag{9}$$

The statements of (8) follow directly from (7) and (6) together with (2). Once we have shown $d \curvearrowleft c \curvearrowleft b$, the other statements of (9) follow from (2).

If not $d \curvearrowleft c \curvearrowleft b$, then (1) would imply $c \curvearrowleft d \curvearrowleft b$ and so (2) would imply $b \curvearrowleft c \curvearrowleft d$. This with $c \curvearrowleft a \curvearrowleft d$ (from (8)) would from (6) imply $b \curvearrowleft a \curvearrowleft d$ which contradicts (8).

(3.3) The Interval Topology Let S be a cycle as above. For any doubleton $\{a,b\}$ comprehended by S we defined in (3.2.5)

$$(\overset{\frown}{b,a}) \equiv \{x\in S : b\frown x\frown a.\} \tag{1}$$

It is a consequence of the foregoing that the sets $(\overset{\frown}{b,a})$ form a base for a topology on the cycle called the **interval topology**.

For a doubleton $\{a,d\}$ subset of S, one may speak of elements x and y of $S\triangle\{a,d\}$ being **on opposite sides of a and d** if

$$\{a,d\}\cap(\overset{\frown}{c,b}) \text{ is a singleton} \tag{2}$$

and **on the same side of a and d** otherwise. It is not difficult to see that the following statements are pairwise equivalent:

$$a \text{ and } c \text{ are on the same side of } b \text{ and } d; \tag{3}$$

$$b \text{ and } d \text{ are on the same side of } a \text{ and } c. \tag{4}$$

(3.4) Lemma I Let $\frown$ be a cyclic order for a set S. Let $\{a,b,c\}$ be a tripleton comprised by S such that $c\frown b\frown a$. The the family

$$\{\{a\},\{b\},\{c\},(\overset{\frown}{b,a}),(\overset{\frown}{c,b}),(\overset{\frown}{a,c})\} \text{ is a partition of } S. \tag{1}$$

This is a consequence of the three axioms of a cycle.

(3.5) Heuristics To intuit the connection between projective space and cyclical order we examine the circle $\mathbb{T}$ as a subset of a plane $\mathbb{P}$.

Each point p of $\mathbb{P}$ corresponds to an involution π of $\mathbb{T}$ which sends each point x of $\mathbb{T}$ to the other point on the intersection of $\mathbb{T}$ with the line passing through both x and p (or to x itself if the line is tangent to $\mathbb{T}$). If the point p is in the disk bounded by the circle, that involution will have no fixed points, and if it is outside the circle, it will have two fixed points.

Now let a and b be two distinct points on $\mathbb{T}$, and let L be the line $\overset{\longleftrightarrow}{a,b}$ in P. The involutions of $\mathbb{T}$ which interchange a with b are just those with correspond to points on L. The line L splits $\mathbb{T}$ into two separate arcs. The involutions corresponding to points surrounded by the circle interchange the arcs, while the involutions corresponding to points outside the circle leave each arc invariant. Furthermore, if m is a point in $\mathbb{T}$ distinct from a and b, then the arc containing m is precisely the set of all $\pi(m)$ such that π corresponds to a point without the circle, or equivalently, the set of all $\pi(m)$ such that π is a homographic involution which has fixed points.

We know (*cf.* (2.9.1)) that in any meridian each homography ϕ can be expressed as a composition of involutions (in fact, of no more than two if need be). The involutions in such a composition of a homography ϕ of $\mathbb{T}$ invert the order of $\mathbb{T}$ if they have fixed points and they preserve it if they do not have fixed points. Thus, if ϕ has a composition consisting of an even number of involutions having fixed points, then **every** composition of ϕ will have an even number of involutions having fixed points: in this case, homographies are either of two types: order preserving or order reversing.

These observations suggest that, for a general meridian M, we separate its homography group into two parts:

$$\mathcal{M}' \equiv \{\phi\in\mathcal{M} : \phi \text{ is a composition of an even number of involutions having fixed points}\} \tag{1}$$

and

$$\mathcal{M}'' \equiv \{\phi\in\mathcal{M} : \phi \text{ is a composition an odd number of involutions having fixed points}\}. \tag{2}$$

If M is to have a cyclic order induced by $\mathcal{M}$, we should have

$$\mathcal{M}' \cap \mathcal{M}'' = \varnothing, \tag{3}$$

The statement (3) is not true for all meridians. For instance, if M is 1-dimensional complex projective space, we may take M to be the union of the complex field C and an additional point ∞. We use the terminology of (2.2.2) and note that $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = \left[\begin{smallmatrix} i & -i \\ i & -i \end{smallmatrix}\right]$, $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = \left[\begin{smallmatrix} 1 & -1 \\ 1 & -1 \end{smallmatrix}\right]$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = \left[\begin{smallmatrix} 0 & \infty \\ 0 & \infty \end{smallmatrix}\right]$. Direct computation shows that $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \circ \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, whence follows that $\left[\begin{smallmatrix} 0 & \infty \\ 0 & \infty \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} i & -i \\ i & -i \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & -1 \\ 1 & -1 \end{smallmatrix}\right]$. Thus $\left[\begin{smallmatrix} 1 & -1 \\ 1 & -1 \end{smallmatrix}\right]$ is in $\mathcal{M}' \cap \mathcal{M}''$.

(3.6) Cyclic Meridians If (3.5.3) holds, then we may define

$$(\forall \{[a,b,c],[x,y,z]\} \subset M_3) \quad [a,b,c] \sim [x,y,z] \iff \left[\begin{smallmatrix} x & y & z \\ a & b & c \end{smallmatrix}\right] \in \mathcal{M}' \tag{1}$$

and $\sim$ will be an equivalence relation with two equivalence classes.

The identity homography is the composition of any involution with itself and we know that each meridian has involutions with fixed points. It follows that the identity is in $\mathcal{M}'$, whence follows that, for each element $[a,b,c]$ of M^3, $[a,b,c] \sim [a,b,c]$. That $\mathcal{M}'$ is closed under compositions, is trivial. Thus, for $\{[a,b,c],[d,e,f],[g,h,i]\} \subset M^3$ such that $[a,b,c] \sim [d,e,f]$ and $[d,e,f] \sim [g,h,i]$, since both $\left[\begin{smallmatrix} d & e & f \\ a & b & c \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} g & h & i \\ d & e & f \end{smallmatrix}\right]$ are in $\mathcal{M}'$, $\left[\begin{smallmatrix} g & h & i \\ d & e & f \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} d & e & f \\ a & b & c \end{smallmatrix}\right]$ must be in $\mathcal{M}'$. But $\left[\begin{smallmatrix} g & h & i \\ d & e & f \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} d & e & f \\ a & b & c \end{smallmatrix}\right]$ is just $\left[\begin{smallmatrix} g & h & i \\ a & b & c \end{smallmatrix}\right]$, whence follows that $[a,b,c] \sim [g,h,i]$. We have established that $\sim$ is an equivalence relation.

Since there exist involutions with fixed points (*cf.* (2.8.4)), $\mathcal{M}''$ is non-void. Hence $\sim$ has more than one equivalence class.

If $\sim$ is a cyclic order, we shall say that M is **cyclic meridian**.

If M is a cyclic meridian and ϕ an element of $\mathcal{M}$, it is evident that the following holds:

$$\phi \in \mathcal{M}' \iff \phi \text{ preserves order for one element of } M^3 \iff \phi \text{ preserves order for all elements of } M^3 \tag{2}$$

and

$$\phi \in \mathcal{M}'' \iff \phi \text{ reverses order for one element of } M^3 \iff \phi \text{ reverses order for all elements of } M^3. \tag{3}$$

(3.7) Theorem I Let M be a meridian such that (3.5.3) holds, where $\mathcal{M}'$and $\mathcal{M}''$are as defined as in (3.5.1) and (3.5.2). Then a necessary and sufficient condition for M to be a cyclic meridian is the following:

$$(\forall \{a,b,c,d\} \subset \mathsf{M} \text{ a quadrupleton}) \quad \left\{\left[\begin{smallmatrix} b & d \\ a & c \end{smallmatrix}\right], \left[\begin{smallmatrix} c & d \\ a & b \end{smallmatrix}\right], \left[\begin{smallmatrix} d & c \\ a & b \end{smallmatrix}\right]\right\} \text{ has precisely one element with no fixed point.} \tag{1}$$

First we suppose that M is a cyclic meridian and consider a general quadrupleton $\{a,b,c,d\}$ comprised by M. By relabelling if necessary, we may suppose that

$$c \frown b \frown a. \tag{i}$$

Then d is in one of the three intervals provided by (3.4.1). Again relabelling if necessary, we may suppose that d is in $(\overset{\frown}{a},c)$: that

$$d \frown c \frown a. \tag{ii}$$

Since $b \frown a \frown c$, it follows from (3.2.9) that

$$d \frown c \frown b. \tag{iii}$$

From $a \frown d \frown c$ and $d \frown c \frown b$ follows that the homography $\left[\begin{smallmatrix} a & d & c \\ d & c & b \end{smallmatrix}\right]$ preserves the order of $d \frown c \frown b$. From (3.6.2) follows that $\left[\begin{smallmatrix} a & d & c \\ d & c & b \end{smallmatrix}\right]$ is in $\mathcal{M}'$. Direct calculation shows that $\left[\begin{smallmatrix} a & d & c \\ d & c & b \end{smallmatrix}\right] = \left[\begin{smallmatrix} c & d \\ a & d \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} b & d \\ a & c \end{smallmatrix}\right]$. Since $\left[\begin{smallmatrix} a & d & c \\ d & c & b \end{smallmatrix}\right]$, being in $\mathcal{M}'$, is a composition of an even number of involutions with fixed points, and since $\left[\begin{smallmatrix} c & d \\ a & d \end{smallmatrix}\right]$ fixes points, the involution $\left[\begin{smallmatrix} b & d \\ a & c \end{smallmatrix}\right]$ must also fix points,

and so

$$\left[\begin{smallmatrix} b & d \\ a & c \end{smallmatrix}\right] \in \mathcal{M}''. \tag{iv}$$

From (i), (ii) and (3.2.8) follows that

$$d \curvearrowleft b \curvearrowleft a. \tag{v}$$

From this and (iv) follows that $\left[\begin{smallmatrix} d & a & b \\ c & d & a \end{smallmatrix}\right]$ preserves the order of $a \curvearrowleft d \curvearrowleft c$. From (3.6.2) follows that $\left[\begin{smallmatrix} d & a & b \\ c & d & a \end{smallmatrix}\right]$ is in $\mathcal{M}'$. Direct calculation shows that $\left[\begin{smallmatrix} d & a & b \\ c & d & a \end{smallmatrix}\right] = \left[\begin{smallmatrix} b & a \\ d & a \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} d & c \\ a & b \end{smallmatrix}\right]$. Since $\left[\begin{smallmatrix} d & a & b \\ c & d & a \end{smallmatrix}\right]$, being in $\mathcal{M}'$, is a composition of an even number of involutions with fixed points, and since $\left[\begin{smallmatrix} b & a \\ d & a \end{smallmatrix}\right]$ fixes points, the involution $\left[\begin{smallmatrix} d & c \\ a & b \end{smallmatrix}\right]$ must also fix points, and so

$$\left[\begin{smallmatrix} d & c \\ a & b \end{smallmatrix}\right] \in \mathcal{M}''. \tag{vi}$$

It follows from (iii) and (v) that $\left[\begin{smallmatrix} b & d & c \\ a & b & d \end{smallmatrix}\right]$ reverses the order of $d \curvearrowleft b \curvearrowleft a$, and so must be in $\mathcal{M}''$. Direct computation shows that $\left[\begin{smallmatrix} b & d & c \\ a & b & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} c & d \\ b & d \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} c & d \\ a & b \end{smallmatrix}\right]$. Since $\left[\begin{smallmatrix} c & d \\ b & d \end{smallmatrix}\right]$ fixes points, it is in $\mathcal{M}''$. Since $\left[\begin{smallmatrix} b & d & c \\ a & b & d \end{smallmatrix}\right]$ is in $\mathcal{M}''$, the involution $\left[\begin{smallmatrix} c & d \\ a & b \end{smallmatrix}\right]$ cannot be. Hence

$$\left[\begin{smallmatrix} c & d \\ a & b \end{smallmatrix}\right] \in \mathcal{M}'. \tag{vii}$$

It follows from (iv), (vi) and (vii) that (1) holds.

Now we suppose that (3.5.3) and (1) hold and show that M is a cyclic projective space: that (3.2.1), (3.2.2) and (3.2.6) hold. We fix an element $[p,q,r]$ of M^3. We shall write that $[x,y,z] \sim [p,q,r]$ whenever there exists an element ϕ of $\mathcal{M}'$ such that $x = \phi(p)$, $y = \phi(q)$ and $z = \phi(r)$. We write $\curvearrowleft$ for the set of all $[x,y,z]$ in M^3 such that $[x,y,z] \sim [p,q,r]$. We shall write $z \curvearrowleft y \curvearrowleft x$ to express that $[x,y,z] \sim [p,q,r]$. It is a consequence of (3.6.2) that $\sim$ is an equivalence relation. Suppose that $[a,b,c]$ is such that

$$c \curvearrowleft b \curvearrowleft a. \tag{viii}$$

Thus $\left[\begin{smallmatrix} a & b & c \\ p & q & r \end{smallmatrix}\right]$ is in $\mathcal{M}'$. Since $\left[\begin{smallmatrix} b & c \\ a & c \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} q & r \\ p & r \end{smallmatrix}\right]$ are involutions with fixed points, $\left[\begin{smallmatrix} b & c \\ a & c \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & b & c \\ p & q & r \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} q & r \\ p & r \end{smallmatrix}\right]$ must be in $\mathcal{M}'$. Direct computation shows that $\left[\begin{smallmatrix} b & c \\ a & c \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} c & a & b \\ p & q & r \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} q & r \\ p & r \end{smallmatrix}\right] = \left[\begin{smallmatrix} b & c & a \\ p & q & r \end{smallmatrix}\right]$, whence follows that $a \curvearrowleft c \curvearrowleft b$: that

$$(3.2.2) \text{ holds.} \tag{ix}$$

Since $\left[\begin{smallmatrix} a & b & c \\ p & q & r \end{smallmatrix}\right]$ is in $\mathcal{M}'$and $\left[\begin{smallmatrix} b & c \\ a & c \end{smallmatrix}\right]$ (being an involution with fixed points) is in $\mathcal{M}''$, it follows that $\left[\begin{smallmatrix} b & c \\ a & c \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & b & c \\ p & q & r \end{smallmatrix}\right]$ is in $\mathcal{M}''$. Direct computation shows that $\left[\begin{smallmatrix} b & c \\ a & c \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} a & b & c \\ p & q & r \end{smallmatrix}\right] = \left[\begin{smallmatrix} b & a & c \\ p & q & r \end{smallmatrix}\right]$. Consequently $[p,q,r] \nsim [b,a,c]$:

$$(3.2.1) \text{ holds.} \tag{x}$$

In addition to (viii), we shall now suppose that

$$b \curvearrowleft x \curvearrowleft a \text{ for some } x \text{ in } M. \tag{xi}$$

It follows from (viii) and (xi) that $[a,b,c] \nsim [c,b,a]$ and so (xi) implies that $[c,b,a] \nsim [a,x,b]$.Consequently $\left[\begin{smallmatrix} c & b & a \\ a & x & b \end{smallmatrix}\right]$ is in $\mathcal{M}''$. Direct computation shows that $\left[\begin{smallmatrix} c & b & a \\ a & x & b \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & c \\ a & b \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} b & x \\ a & c \end{smallmatrix}\right]$. Since $\left[\begin{smallmatrix} a & c \\ a & b \end{smallmatrix}\right]$ is in $\mathcal{M}''$, the involution $\left[\begin{smallmatrix} c & b \\ x & a \end{smallmatrix}\right]$ must be in $\mathcal{M}'$. From (1) follows that $\left[\begin{smallmatrix} a & b \\ x & c \end{smallmatrix}\right]$ is in $\mathcal{M}''$. Since $\left[\begin{smallmatrix} x & b \\ a & b \end{smallmatrix}\right]$ is in $\mathcal{M}''$, the composition $\left[\begin{smallmatrix} a & b \\ x & c \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} x & b \\ a & b \end{smallmatrix}\right]$ is in $\mathcal{M}'$. Direct computation shows that $\left[\begin{smallmatrix} a & b \\ x & c \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} x & b \\ a & b \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & x & c \\ a & x & b \end{smallmatrix}\right]$. From (ix) follows that $[a,x,c] \in \curvearrowleft$:

$$(3.2.6) \text{ holds.} \tag{xii}$$

It follows from (ix), (x) and (xii) that M is a cyclic meridian.

(3.8) Lemma II Let $\curvearrowleft$ be a cyclic order on a cyclic meridian M and let $\{a,b,c\}$ be a tripleton such that $c \curvearrowleft b \curvearrowleft a$. Then

$$(\overset{\frown}{c,a}) = \{\left[\begin{smallmatrix} c & x \\ a & x \end{smallmatrix}\right](b) : x \in (M \triangle \{a,c\})\}. \tag{1}$$

Let t be any element of $(\overset{\frown}{c,a})$. It follows that $\left[\begin{smallmatrix} a & t & c \\ a & b & c \end{smallmatrix}\right]$ is in $\mathcal{M}'$. Since $\left[\begin{smallmatrix} a & t & c \\ a & b & c \end{smallmatrix}\right] = \left[\begin{smallmatrix} c & t \\ a & t \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} c & t \\ a & b \end{smallmatrix}\right]$, it follows that $\left[\begin{smallmatrix} c & t \\ a & b \end{smallmatrix}\right]$ must have some fixed point x: $\left[\begin{smallmatrix} c & t \\ a & b \end{smallmatrix}\right] = \left[\begin{smallmatrix} c & x \\ a & x \end{smallmatrix}\right]$. Consequently $t = \left[\begin{smallmatrix} a & t & c \\ a & b & c \end{smallmatrix}\right](b) = \left[\begin{smallmatrix} c & t \\ a & t \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} c & x \\ a & x \end{smallmatrix}\right](b)$, which implies

that $\left[\begin{smallmatrix} c & x \\ a & x \end{smallmatrix}\right](b)$ must be t. We have shown that

$$(\overset{\frown}{c,a}) \subset \{\left[\begin{smallmatrix} c & x \\ a & x \end{smallmatrix}\right](b) : x \in (M \triangle \{a,c\})\} \tag{i}$$

We now let x be any element of M distinct from a and c. Let $t \equiv \left[\begin{smallmatrix} c & x \\ a & x \end{smallmatrix}\right](b)$. Then $\left[\begin{smallmatrix} a & t & c \\ a & b & c \end{smallmatrix}\right] = \left[\begin{smallmatrix} c & t \\ a & t \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} c & x \\ a & x \end{smallmatrix}\right]$ and the right-hand side is in $\mathcal{M}'$. Thus $\left[\begin{smallmatrix} a & t & c \\ a & b & c \end{smallmatrix}\right]$ is in $\mathcal{M}'$and so $c \frown t \frown a$: i.e. $t \in (\overset{\frown}{c,a})$. We have shown $\{\left[\begin{smallmatrix} c & x \\ a & x \end{smallmatrix}\right](b) : x \in (M \triangle \{a,c\})\} \subset (\overset{\frown}{c,a})$, which, with (i), yields (1).

(3.9) Lemma III Suppose that $\{a,b,c,d\}$ is a quadrupleton comprised by a cyclic meridian M, that a and b are on the same side of c and d and that ϕ is an element of $\mathcal{M}$. Then $\phi(a)$ and $\phi(b)$ are on the same side of $\phi(c)$ and $\phi(d)$.

It follows from the hypothesis that $\left[\begin{smallmatrix} c & b & d \\ c & a & d \end{smallmatrix}\right]$ is in $\mathcal{M}'$. It is evident that ϕ is in $\mathcal{M}'$ if, and only if, ϕ^{-1} is in $\mathcal{M}'$. It follows that $\phi \circ \left[\begin{smallmatrix} c & b & d \\ c & a & d \end{smallmatrix}\right] \circ \phi^{-1}$ is in $\mathcal{M}'$. But direct computation shows that $\phi \circ \left[\begin{smallmatrix} c & b & d \\ c & a & d \end{smallmatrix}\right] \circ \phi^{-1}$ is just $\left[\begin{smallmatrix} \phi(c) & \phi(b) & \phi(d) \\ \phi(c) & \phi(a) & \phi(d) \end{smallmatrix}\right]$. Hence $\phi(a)$ and $\phi(b)$ are on the same side of $\phi(c)$ and $\phi(d)$.

(3.10) Lemma IV Let $\{a,b,c,d\}$ be a quadrupleton comprised by a cyclic meridian M such that

$$b \text{ and } c \text{ are on the same side of } a \text{ and } d \tag{1}$$

and such that

$$d \frown c \frown a. \tag{2}$$

Then

$$\left[\begin{smallmatrix} d & c \\ d & b \end{smallmatrix}\right](d) \text{ and } b \text{ are on the same side of } a \text{ and } d. \tag{3}$$

We first note that, given (1), (2) is equivalent to $d \frown b \frown a$ and (3) is equivalent to both $\left[\begin{smallmatrix} a & c \\ a & b \end{smallmatrix}\right](d)$ and c being on the same side of a and d. Consequently, interchanging the elements b and and c would not alter the truth or falsity of Lemma IV. Thus we may and shall make the additional presumption that

$$c \frown b \frown a. \tag{i}$$

It follows from (i) and (2) that all the relations of (3.2.8) and (3.2.9) hold.

Let e denote $\left[\begin{smallmatrix} d & c \\ d & b \end{smallmatrix}\right](a)$ and assume that

$$e \text{ and } b \text{ were not on the same side of } a \text{ and } d. \tag{ii}$$

From (1) and (2) follows that $d \frown b \frown a$ which, with the Cyclic Axiom II, yields $b \frown a \frown d$. From this and (ii) follows that $e \frown a \frown d$ could not be, whence from the Cyclic Axiom I follows that

$$a \frown e \frown d. \tag{iii}$$

From (1) and (ii) follows that

$$e \text{ and } c \text{ could not be on the same side of } a \text{ and } d. \tag{iv}$$

The homography $\left[\begin{smallmatrix} d & c \\ d & b \end{smallmatrix}\right]$ has a fixed point and so reverses the order of ordered triples. In particular, if p is the fixed point of $\left[\begin{smallmatrix} d & c \\ d & b \end{smallmatrix}\right]$ distinct from d, $\left[\begin{smallmatrix} d & c \\ d & b \end{smallmatrix}\right]$ sends $(\overset{\frown}{p,d})$ to $(\overset{\frown}{d,p})$ and *vice versa*. If b and c were on the same side of d and p, since $\left[\begin{smallmatrix} d & c \\ d & b \end{smallmatrix}\right]$ leaves $\{b,c\}$ invariant, the element of $\{(\overset{\frown}{p,d}),(\overset{\frown}{d,p})\}$ containing b and c would be sent to itself. It follows that

$$b \text{ and } c \text{ are on opposite sides of } d \text{ and } p. \tag{v}$$

That $d \frown c \frown b$ follows from (3.2.9). This with (v) implies that $c \frown p \frown b$. From this and $d \frown c \frown b$ follows via the Cyclic Axiom III that

$$d \frown p \frown b. \tag{vi}$$

It follows from (3.2.8) that

$$b \frown a \frown d. \tag{vii}$$

From (iv) and (vii) follows that $e \frown a \frown d$ could not be: the Cyclic Axiom I implies

$$a \frown e \frown d. \tag{viii}$$

We recall that the homography $\left[\begin{smallmatrix} d & c \\ d & b \end{smallmatrix}\right]$ sends $(\overset{\frown}{p,d})$ to $(\overset{\frown}{d,p})$ and *vice versa*. If a and e were on the same side of d and p, since $\left[\begin{smallmatrix} d & c \\ d & b \end{smallmatrix}\right]$ leaves $\{a,e\}$ invariant, the element of $\{(\overset{\frown}{p,d}),(\overset{\frown}{d,p})\}$ containing a and e would be sent to itself. It follows that

$$a \text{ and } e \text{ are on opposite sides of } d \text{ and } p. \tag{ix}$$

In view of (viii), it follows from (ix) that $a \frown e \frown p$ could not hold: the Cyclic Axiom I implies that $a \frown p \frown e$. From the Cyclic Axiom II follows that $p \frown e \frown a$. From the Cyclic Axiom II and (viii) follows that $e \frown d \frown a$. These two $p \frown e \frown a$ and $e \frown d \frown a$ together with the Cyclic Axiom III imply that $p \frown d \frown a$. Then from the Cyclic Axiom II comes that $a \frown p \frown d$. Applying the Cyclic Axiom III to this and (vii) yields

$$b \frown p \frown d. \tag{x}$$

Statements (vi) and (x) are inconsistent. It follows that (ii) cannot be: that (3) must hold.

(3.11) Theorem II Let M be a meridian. Necessary and sufficient conditions for M to be a real meridian are the following

$$M \text{ is a cyclic meridian;} \tag{1}$$

and, if $\mathfrak{M}$ is the cyclic topology on M,

$$M \text{ is compact relative to } \mathfrak{M}. \tag{2}$$

A real meridian is isomorphic to a circle, and hence homeomorphic to a circle with its arc topology. It follows that (1) and (2) hold. Suppose on the other hand that (1) and (2) hold.

Let $\{\infty,o,l\}$ be any tripleton in M. Let F denote the associated field, as in (2.8.10) . Since M is a cyclic meridian, we may consider the cyclic order $\frown$ for which $\infty \frown l \frown o$. We shall show that the open interval $(\overset{\frown}{\infty,o})$ is a positive cone for F in the sense of (Real and Abstract Analysis.5.7), that F is Archimedean ordered in the sense of (Real and Abstract Analysis.5.17) and that it is order complete in the sense of (Real and Abstract Analysis.5.25). It will then follow from (Real and Abstract Analysis.5.34) that F is isomorphic to the field of real numbers and so M is a real meridian.

We know that $F = \{o\} \cup (\overset{\frown}{\infty,o}) \cup (\overset{\frown}{o,\infty})$, that $(\overset{\frown}{\infty,o}) \cap (\overset{\frown}{o,\infty}) = \varnothing$ and that $(\overset{\frown}{o,\infty}) = \{-x : x \in (\overset{\frown}{o,\infty})\}$. Suppose that x and y are elements of $(\overset{\frown}{\infty,o})$. Thus as x and y are on the same side of o and ∞. We apply (3.10) by setting $a \equiv o$, $b \equiv x$, $c \equiv y$ and $d \equiv \infty$. It follows from (3.10) that $\left[\begin{smallmatrix} \infty & x \\ \infty & y \end{smallmatrix}\right](o)$ and x are on the same side of o and ∞. But $x+y = \left[\begin{smallmatrix} \infty & x \\ \infty & y \end{smallmatrix}\right](o)$. Hence $x+y$ is in $(\overset{\frown}{o,\infty})$. Since $x \cdot y = \left[\begin{smallmatrix} o & x \\ \infty & y \end{smallmatrix}\right](l)$, it follows from (3.8) that $x \cdot y$ is in $(\overset{\frown}{o,\infty})$. We have established that $(\overset{\frown}{o,\infty})$ is a positive cone for F.

Assume that F were not Archimedean ordered relative to $(\overset{\frown}{o,\infty})$. Then there would exist p and q in $(\overset{\frown}{o,\infty})$ such that the sequence $(p,p+1,p+2,\dots)$ were in $(\overset{\frown}{o,q})$. Since M is compact, it would have a least upper bound u in $(\overset{\frown}{o,\infty})$. The sequence would eventually be in $(\overset{\frown}{u\text{-}l,u})$ and so there would exist a positive integer n such that $n \cdot l + p$ were in $(\overset{\frown}{u\text{-}l,u})$. But then $(n+1) \cdot l + p$ would be in $(\overset{\frown}{u,u+1})$: an absurdity. Thus F is Archimedean ordered.

That F is order complete follows from the fact that M is compact.

(3.12) Corollary I Let $\phi \,|\, M \hookrightarrow N$ be a projective isomorphism from one real meridian onto another. Then ϕ is a homeomorphism.

It will suffice to show that the inverse of a basis for the topology of N is a basis of the topology of M. Let $\{a,b\}$ be a doubleton comprised by N and q any element of $(\overset{\frown}{b,a})$. Then

$$(\phi^{-1}(b),\overset{\frown}{\phi^{-1}(a)}) \overset{\text{by (3.8.1)}}{=\!=} \{\left[\begin{smallmatrix} \phi^{-1}(a) & x \\ \phi^{-1}(b) & x \end{smallmatrix}\right](\phi^{-1}(q)) : x \in M \triangle \{\phi^{-1}(a), \phi^{-1}(a)\}\} =$$

$$\{\begin{matrix}\phi^{-1}(a) & \phi^{-1}(y)\\ \phi^{-1}(b) & \phi^{-1}(y)\end{matrix}(\phi^{-1}(q)) : y \in N \triangle \{a,b\}\} \overset{\text{by (2.7.5)}}{=\!=} \{\phi^{-1}(\begin{matrix}a & y\\ b & y\end{matrix}(q)) : y \in N\} \overset{\text{by (3.8.1)}}{=\!=} \{\phi^{-1}(t) : t \in (\overset{\curvearrowleft}{b,a})\}.$$

(3.13) Corollary II Each projective isomorphism of one real projective space onto another is an isography.

Let $[a,b,c]$ be an ordered tripleton in a real meridian M and let d be a fourth point of M. Let ϕ be a projective isomorphism of M. From (3.12) follows that ϕ is continuous. Since Q is dense in M, there are sequences a, b, c and d in Q which converge to $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d}$ respectively. It follows from (2.14.5) that at each subscript we have $\phi(d) = \begin{matrix}\phi(a) & \phi(b) & \phi(c)\\ a & b & c\end{matrix}(d)$. It follows that

$$\phi(d) = lim\ \phi(d) = lim \begin{matrix}\phi(a) & \phi(b) & \phi(c)\\ a & b & c\end{matrix}(d) = \begin{matrix}\phi(a) & \phi(b) & \phi(c)\\ a & b & c\end{matrix}(d).$$

Consequently ϕ is isographic.

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